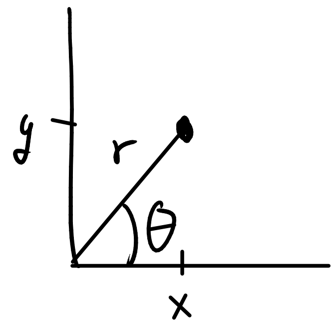
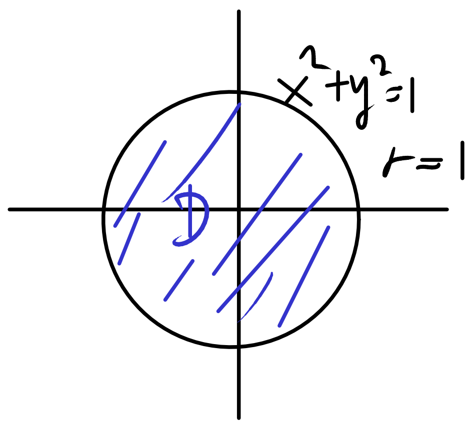
Double integrals in polar coordinates.



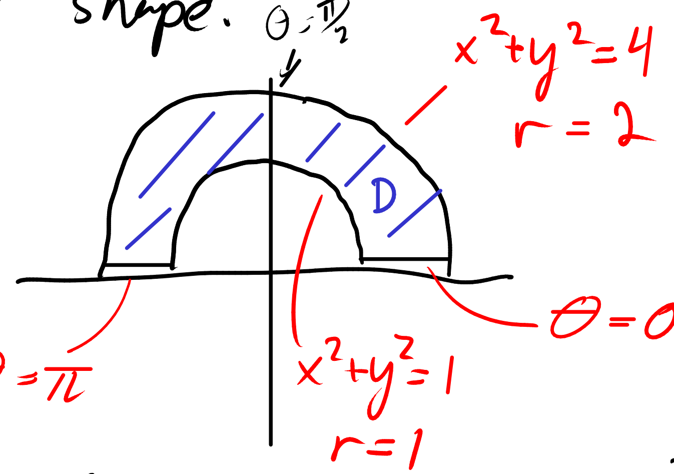
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r^2 = x^2+y^2 \\ \tan\theta = y/x \end{cases}$$

Sometimes polar coordinates are useful for integrals especially when the region of integration D has a "circular shape".



$D = \{ 0 \le r \le 1, 0 \le \theta \le 2\pi \}$

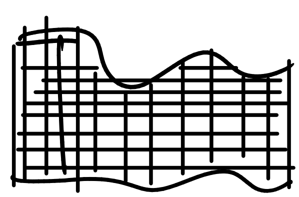


$D = \{ 1 \le r \le 2, \ 0 \le \theta \le \pi \}$

Suppose want to approximate $\iint_D f(x,y)\, dA$

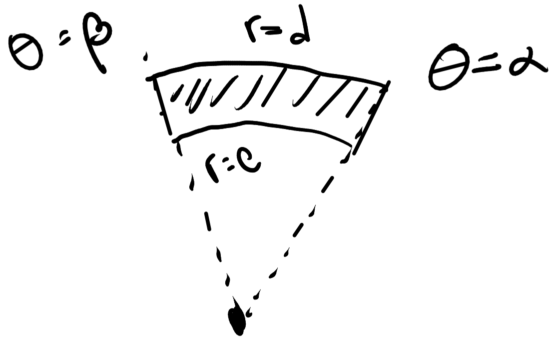
by a Riemann sum adapted to polar coordinates.

Recall rectangular coords (x,y)



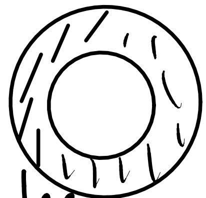
Break into rectangles

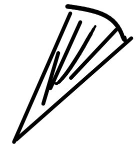
For polar (r,θ) use "polar rectangle"



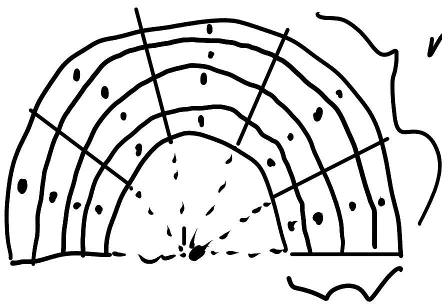
$\{c \le r \le d,\ \alpha \le \theta \le \beta\}$

"annular sector"





Subdivide into polar rectangles



m divisions of θ

n divisions of r

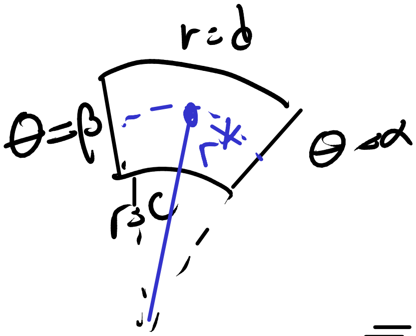
Riemann sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}, y_{ij})\,\Delta A_{ij}$$

value at a point in the subregion

area of subregion.

Q: what is ΔA_{ij} in (r,θ)-variables
What is the area of a polar rectangle?



$$A = \frac{1}{2}d^2(\beta-\alpha) - \frac{1}{2}c^2(\beta-\alpha)$$

$$= \frac{1}{2}(d^2-c^2)\,\Delta\theta$$

$$= \frac{1}{2}(d+c)(d-c)\,\Delta\theta$$

$$= \frac{1}{2}(d+c)\,\Delta r\,\Delta\theta$$

$\Delta\theta = \beta-\alpha$
$\Delta r = d-c$

define $r^* = \frac{1}{2}(d+c)$ Average (or midpoint) of inner and outer radii.

$$A = r^* \Delta r \Delta\theta$$

Write Riemann sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}, y_{ij})\,\Delta A_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}, y_{ij})\, r^*_{ij} \Delta r_{ij} \Delta\theta_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} f(r_{ij}\cos\theta_{ij},\ r_{ij}\sin\theta_{ij})\ r^*_{ij}\,\Delta r_{ij}\,\Delta\theta_{ij}$$

To get to integral: take limit $\begin{cases} n\to\infty, & m\to\infty \\ \Delta r\to 0, & \Delta\theta\to 0. \end{cases}$

$$\to \int_{\theta=\theta_0}^{\theta=\theta_1}\int_{r=r_0}^{r=r_1} f(r\cos\theta,\ r\sin\theta)\,\boxed{r}\,dr\,d\theta$$

v. important.

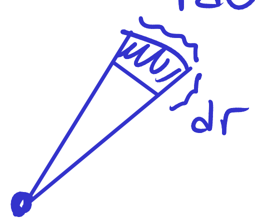
To convert $\iint_D f(x,y)\,dA$ to $\int_{\theta_0}^{\theta_1}\int_{r_0}^{r_1} f(r\cos\theta, r\cos\theta)\, r\,dr\,d\theta$

(1) change x and y into $r\cos\theta$ and $r\sin\theta$ inside $f(x,y)$

(2) determine limits $r_0, r_1, \theta_0, \theta_1$ (draw picture)

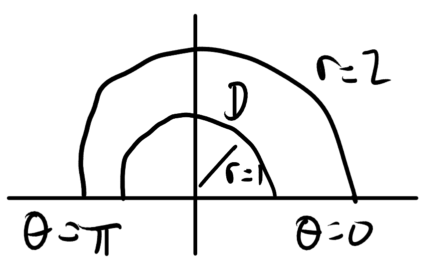
(3) $dA = r\,dr\,d\theta = r\,d\theta\,dr$

To remember (3) very small polar rectangle is similar to a very small rectangular rectangle



$dA = dr \cdot r\,d\theta = r\,dr\,d\theta$

Ex



$$\iint_D (3x+4y^2)\,dA$$

$$3x+4y^2 = 3(r\cos\theta) + 4(r\sin\theta)^2$$
$$= 3r\cos\theta + 4r^2\sin^2\theta$$

$dA = r\,dr\,d\theta$

$$\int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^{2} \left(3r^2\cos\theta + 4r^3\sin^2\theta\right) dr\, d\theta$$

$$= \int_0^{\pi} \left[r^3\cos\theta + r^4\sin^2\theta \right]_{r=1}^{r=2} d\theta$$

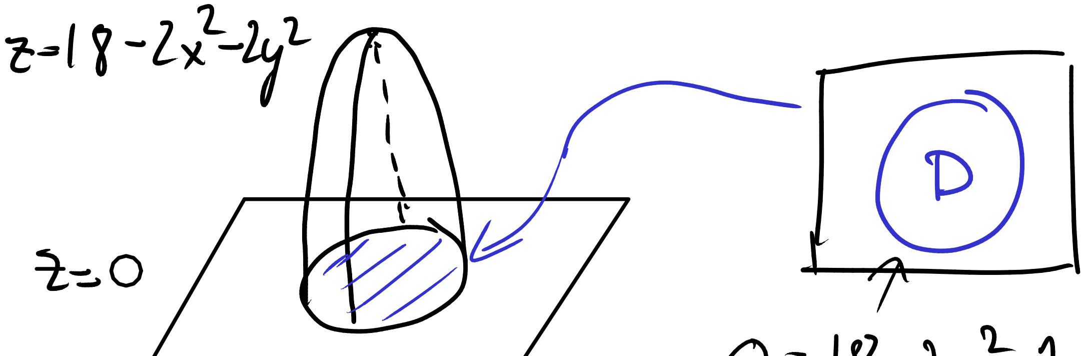
$$= \int_0^{\pi} \left(7\cos\theta + 15\sin^2\theta\right) d\theta$$

$$= \int_0^{\pi} \left(7\cos\theta + 15\left(\frac{1-\cos 2\theta}{2}\right)\right) d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

$$= \frac{15}{2}\pi$$

Ex: Volume of region bounded by $z = 18 - 2x^2 - 2y^2$ and $z = 0$



$$0 = 18 - 2x^2 - 2y^2$$

$$2x^2 + 2y^2 = 18$$
$$x^2 + y^2 = 9$$

circle of radius 3

D

$$\text{Vol} = \iint_D (18 - 2x^2 - 2y^2)\, dA \qquad D = \{0 \le r \le 3, 0 \le \theta \le 2\pi\}$$

$$18 - 2x^2 - 2y^2 = 18 - 2(r\cos\theta)^2 - 2(r\sin\theta)^2$$
$$= 18 - 2r^2\cos^2\theta - 2r^2\sin^2\theta$$
$$= 18 - 2r^2(\cos^2\theta + \sin^2\theta)$$
$$= 18 - 2r^2$$

Or $18 - 2x^2 - 2y^2 = 18 - 2(x^2 + y^2)$

$$(r^2 = x^2 + y^2)$$

$$= 18 - 2r^2$$

$$\int_0^{2\pi}\int_0^3 (18 - 2r^2)\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (18r - 2r^3)\,dr\,d\theta$$

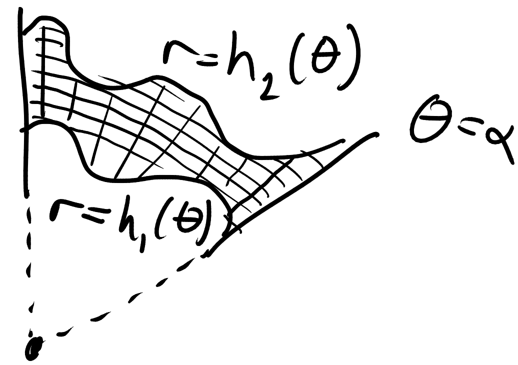
$$= \int_0^{2\pi} \left[9r^2 - \frac{1}{2}r^4\right]_0^3 d\theta = \int_0^{2\pi} \left(81 - \frac{1}{2}81\right) d\theta$$

$$= \int_0^{2\pi} \frac{81}{2}\, d\theta = \frac{81}{2}(2\pi) = 81\pi.$$

Polar coordinates very useful when integrand only depends on r, not θ.

$$\int_0^{2\pi}\int_a^b f(r)\, r\,dr\,d\theta = 2\pi \int_a^b f(r) \cdot r\,dr$$

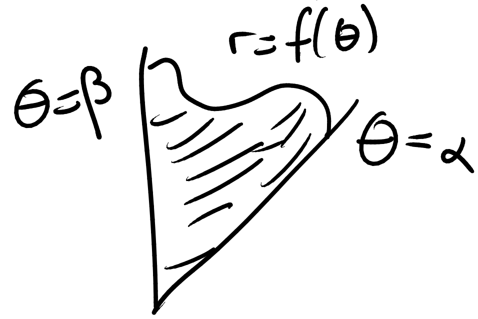
More general regions:



Region between two polar graphs $r=h_1(\theta)$
$r=h_2(\theta)$

$$\iint_D f\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

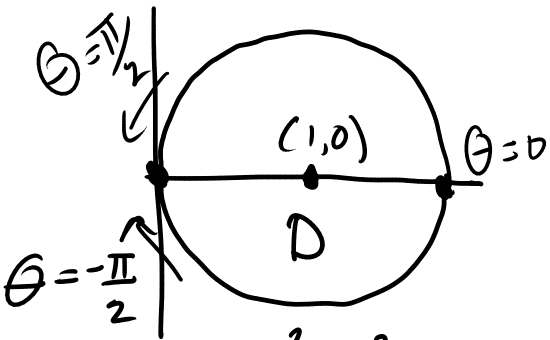
Area inside polar curve $r=f(\theta)$



$$\text{Area} = \iint_D 1\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_0^{r=f(\theta)} r\,dr\,d\theta$$

$$= \int_\alpha^\beta \left[\tfrac{1}{2}r^2\right]_0^{f(\theta)} d\theta = \int_\alpha^\beta \tfrac{1}{2}\,[f(\theta)]^2\,d\theta$$

(remember?)



$$f(x,y) = x^2+y^2 = r^2$$

$$\iint_D (x^2+y^2)\,dA$$

$$(x-1)^2+y^2=1$$

$$r = h(\theta)$$

$$= \iint_D r^2 \, r\,dr\,d\theta$$

$$r = 2\cos\theta$$

$$D = \left\{ 0 \le r \le 2\cos\theta \,,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \right\}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{4}r^4\right]_0^{2\cos\theta} d\theta$$

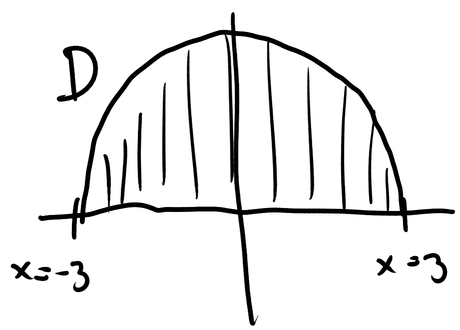
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}(2\cos\theta)^4\,d\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta .$$

(DeMoivre's formula where $i = \sqrt{-1}$

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$)

Common situation: convert iterated integral in (x,y)-coordinates into polar coordinates.

Ex $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$



$y=\sqrt{9-x^2}$

$y^2=9-x^2$

$x^2+y^2=9$ circle of radius 3

Region is a semicircle.

In polar coordinates $D=\{0\le r\le 3,\ 0\le\theta\le\pi\}$

$\sin(x^2+y^2)=\sin(r^2)$

$dy\,dx=dA=r\,dr\,d\theta$

$$\int_0^{\pi}\int_0^{3}\sin(r^2)\,r\,dr\,d\theta$$

$u=r^2 \quad du=2r\,dr$

$$=\int_0^{\pi}\int_0^{9}\sin(u)\,\frac{du}{2}\,d\theta$$

$$= \int_0^{\pi} \left[-\frac{1}{2} \cos u \right]_{u=0}^{u=9} d\theta$$

$$= \int_0^{\pi} \left[-\frac{1}{2} \cos 9 + \frac{1}{2} \cos 0 \right] d\theta$$

$$= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2} \cos 9 \right) d\theta = \frac{\pi}{2} \left(1 - \cos 9 \right)$$