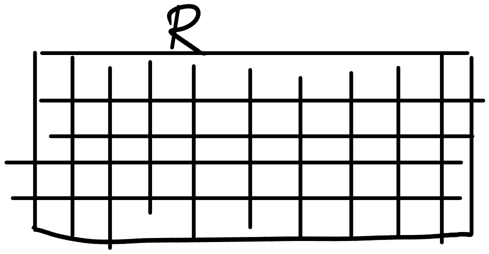
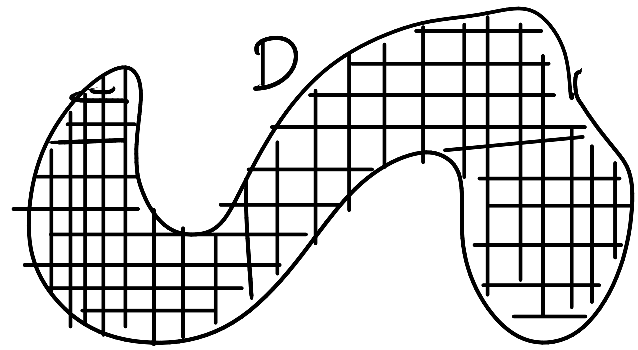
Double integrals over general regions.



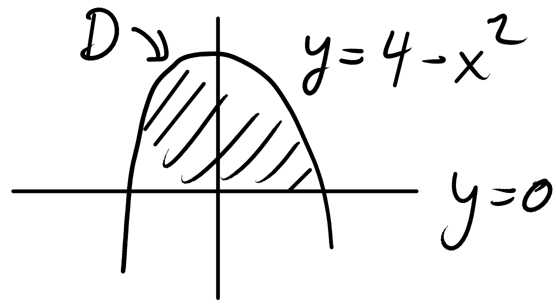


use Riemann sum based on this grid.

want to write

$$\iint_D f(x,y)\,dA$$ as an iterated integral.

<u>Example</u> $$\iint_D (2-x)\,dA$$



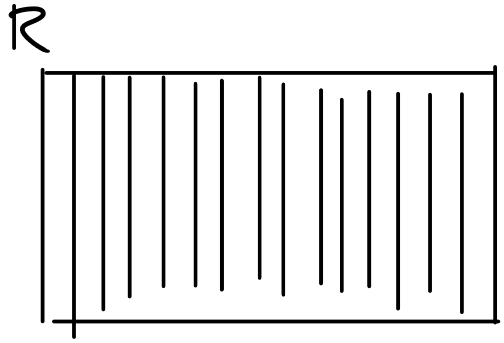
<u>Idea</u>: a double integral is like a sum with two indices

$$\iint f\,dA \approx \sum_i \sum_j f(x_i, y_j)\,\Delta A$$

1	2	4	8	15
1	1	7	2	11
3	4	1	5	13
5	7	12	15	39

Either sum along columns then sum results, or sum along rows first. ...

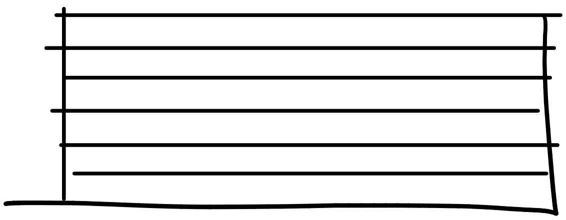
$\iint_R f\,dA$



Integrate along vertical line segments $\int_c^d f\,dy$

Integrate the results $\int_a^b \int_c^d f\,dy\,dx$

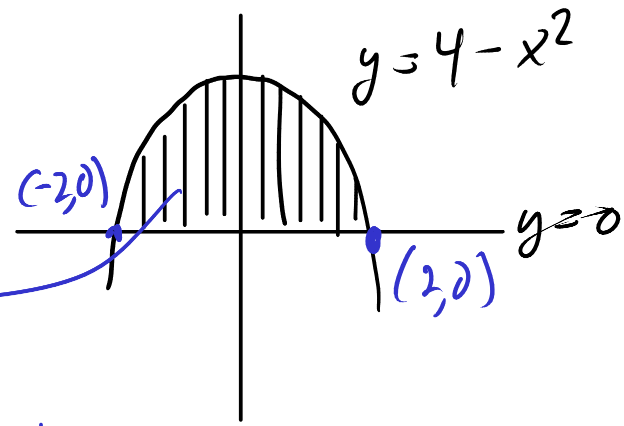
OR:



Integrate along horizontal $\int_a^b f\,dx$

Integrate results $\int_c^d \int_a^b f\,dx\,dy$

$\iint (2-x)\,dA$



this line segment goes from $y=0$ to $y=4-x^2$
(length depends on x)

x goes from $\underline{-2}$ to $\underline{2}$. (min and max over whole region)

Integrate along vertical segments $\displaystyle\int_0^{4-x^2} (2-x)\,dy$

Integrate results: $$\int_{-2}^{2}\int_0^{4-x^2} (2-x)\,dy\,dx$$

$$= \int_{-2}^{2} \Big[(2-x)y\Big]_{y=0}^{y=4-x^2} dx = \int_{-2}^{2} \Big((2-x)(4-x^2)\Big)\,dx$$

$$= \int_{-2}^{2} (8-4x-2x^2+x^3)\,dx = \cdots$$

<u>Note</u>: that x appears in the limit of integration of the inner integral w.r.t. y.

(x cannot appear in the limit of an integral w.r.t. x

$$\int_0^x f\,dx$$

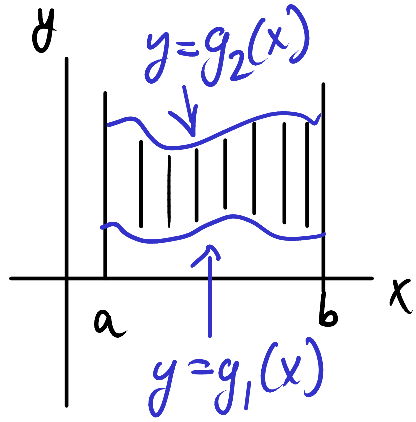
meaningless

The limits of integration on outer integral are just numbers (don't depend on x or y)

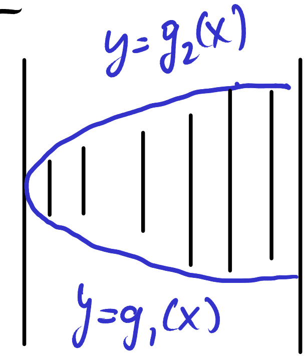
Generalize

"Type I region"

∫ dy first

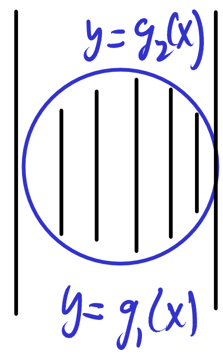


or

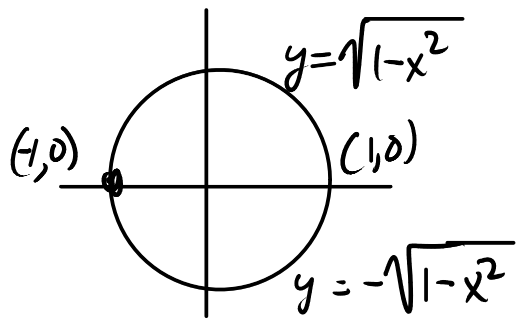


$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

or

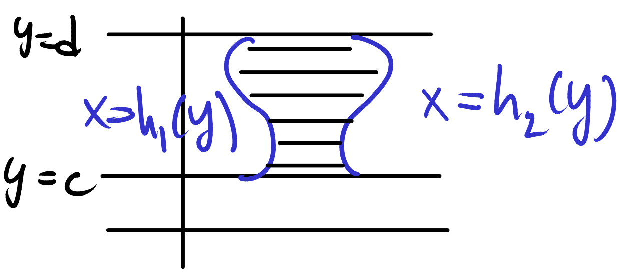


Eg $\quad x^2+y^2=1$



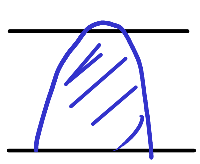
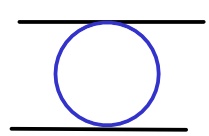
$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

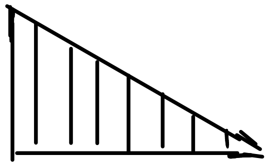
"Type II"



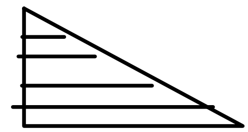
$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

(left to right)

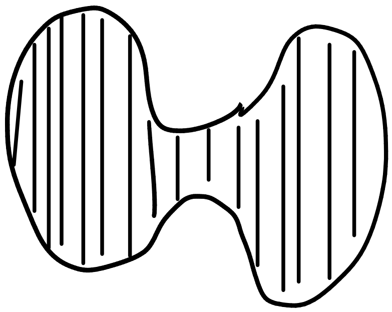
or  or 



or

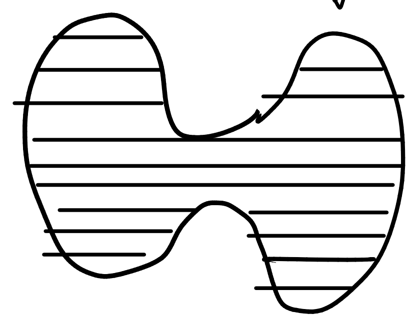


Both Type I and Type II

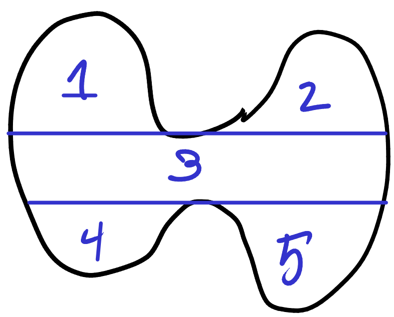


Type I

not Type II

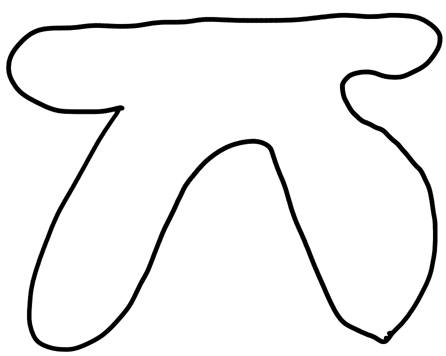


But can break up into type II regions



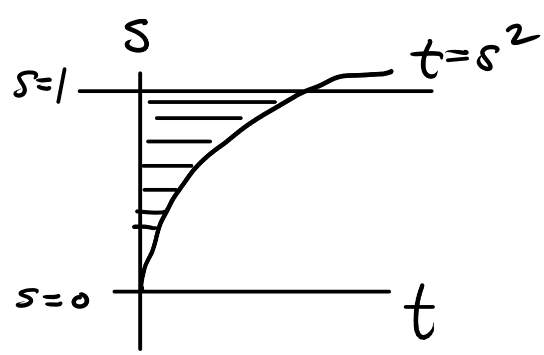
Each of 5 regions is type II.

Neither I nor II



Examples

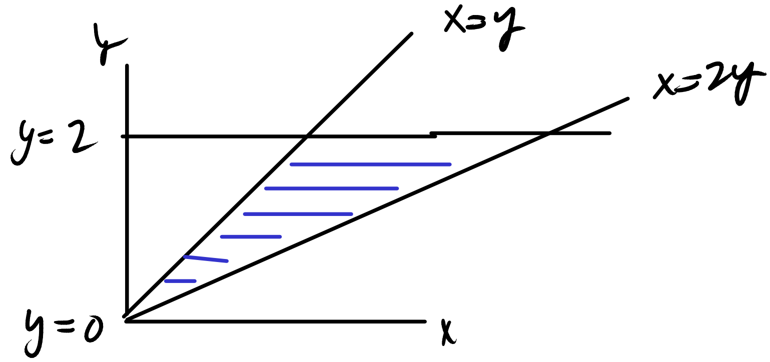
$$\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds$$



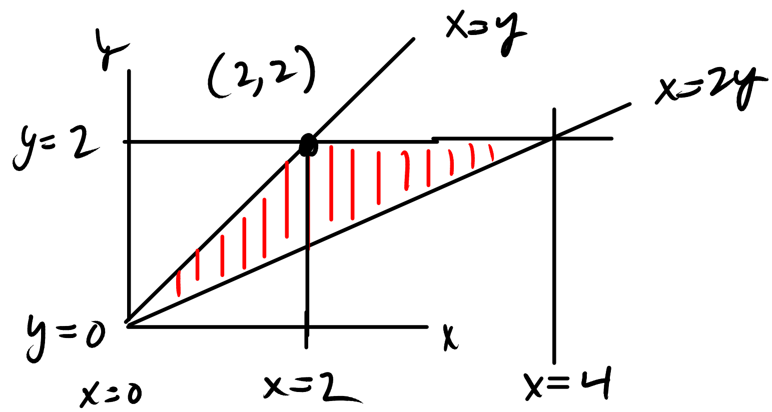
$$\int_0^1 \Big[\cos(s^3)\, t\Big]_{t=0}^{t=s^2} ds = \int_0^1 \cos(s^3)\, s^2\, ds$$

$$\boxed{u = s^3 \quad du = 3s^2 ds} = \int_0^1 \cos(u)\, \tfrac{1}{3}\, du \ \ldots$$

$$\int_0^2 \int_y^{2y} xy\, dx\, dy$$



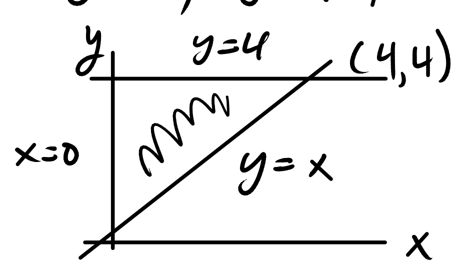
Other order



$$\text{Integral} = \int_{x=0}^{x=2} \int_{y=\frac{1}{2}x}^{y=x} xy\, dy\, dx + \int_{x=2}^{x=4} \int_{y=\frac{1}{2}x}^{y=2} xy\, dy\, dx$$

<u>Example</u> $\iint_D y^2 e^{xy}\, dA$ D is enclosed by $y=x$, $y=4$, $x=0$.

Which order is easier?



Type I $\int dy$ first

$$\int_0^4 \int_x^4 y^2 e^{xy}\, dy\, dx$$

$\downarrow$

integrate by parts twice.

Type II $\int dx$ first

$$\int_0^4 \int_0^y y^2 e^{xy}\, dx\, dy$$

$$\left[\frac{\partial}{\partial x} e^{xy} = e^{xy} \cdot y \right]$$

$$\int_0^4 \left[y^2 \frac{1}{y} e^{xy} \right]_0^y dy$$

$$\left(y e^{xy} \right]_0^y$$

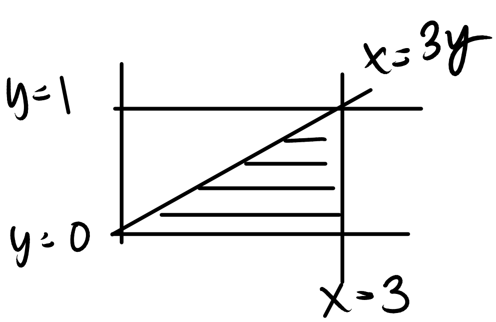
$$\int_0^4 \left[\underbrace{y e^{y^2}}_{} - y \right] dy$$

u sub $u = y^2$ . . .

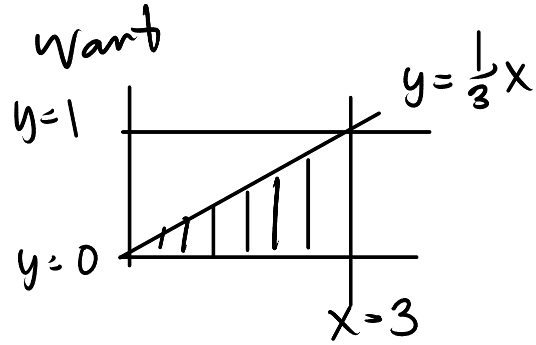
Ex: switch the order (to make the integral easier)

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$$ impossible as written

Diagram:



want



$$\int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^3 \left[y e^{x^2} \right]_{y=0}^{y=\frac{x}{3}} dx = \int_0^3 \frac{x}{3} e^{x^2}\, dx$$

$u = x^2$ . . .

$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\, dx\, dy$$

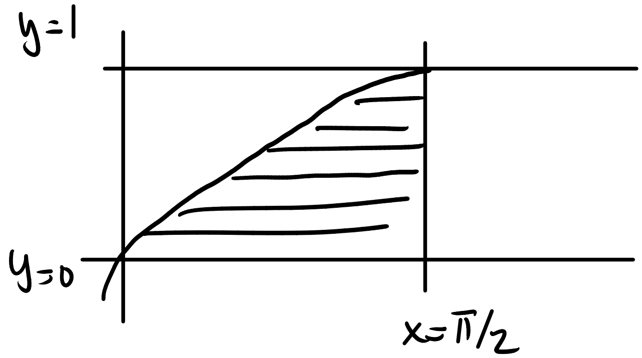
arcsin = $\sin^{-1}$

$y=0$

$y=1$

$x=\pi/2$

$x=\arcsin y \Rightarrow y=\sin x$



other order

$$\int_0^{\pi/2} \int_0^{\sin x} \cos x \sqrt{1+\cos^2 x}\, dy\, dx$$

$$\int_0^{\pi/2} \left[\cos x \sqrt{1+\cos^2 x} \cdot \sin x \right] dx$$

$$= \int_0^{\pi/2} \sqrt{1+\cos^2 x}\, (\cos x \sin x)\, dx$$

$u = 1+\cos^2 x \qquad du = 2\cos x\,(-\sin x)$

$= -2\cos x \sin x$