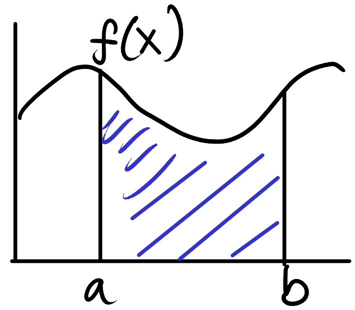
Double integrals

Rectangles and iterated integrals.

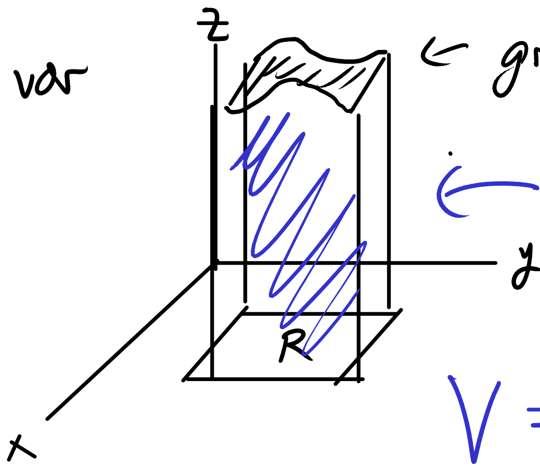
1 var



$$A = \int_a^b f(x)\,dx$$

area under curve

2 var



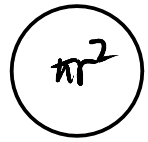
← graph of $f(x,y)$

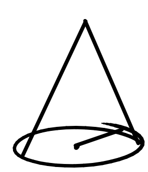
← want to compute this volume

$$V = \iint_R f(x,y)\,dA$$

How to find areas and volumes?

(1) Exact formular

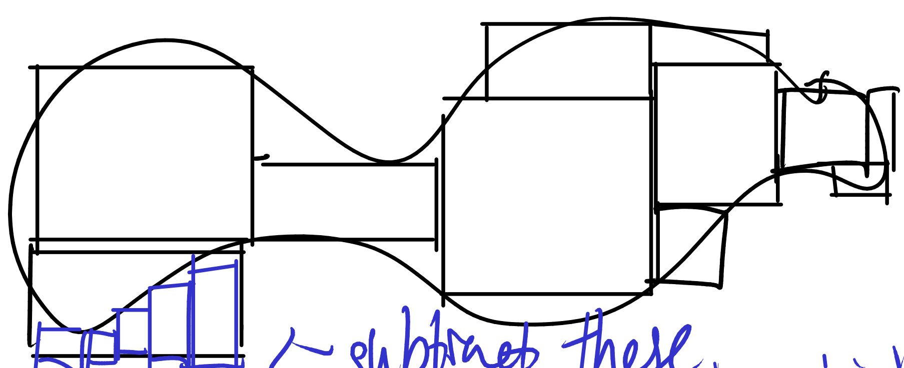




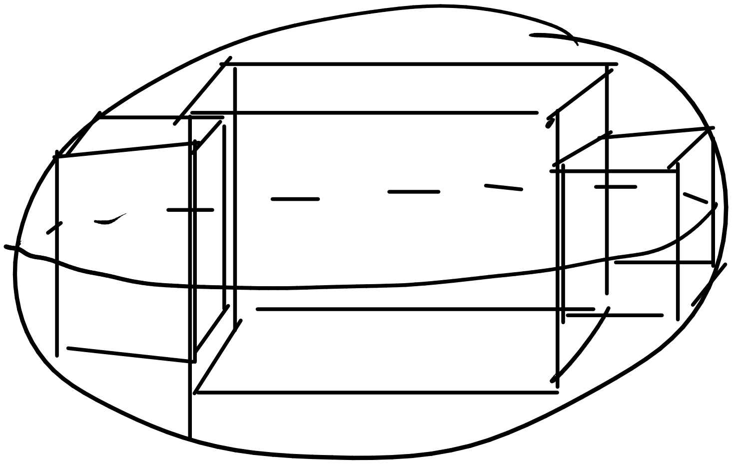
$\frac{1}{3}\pi r^2 h$

(2) Approximate by breaking up the shape

A=?



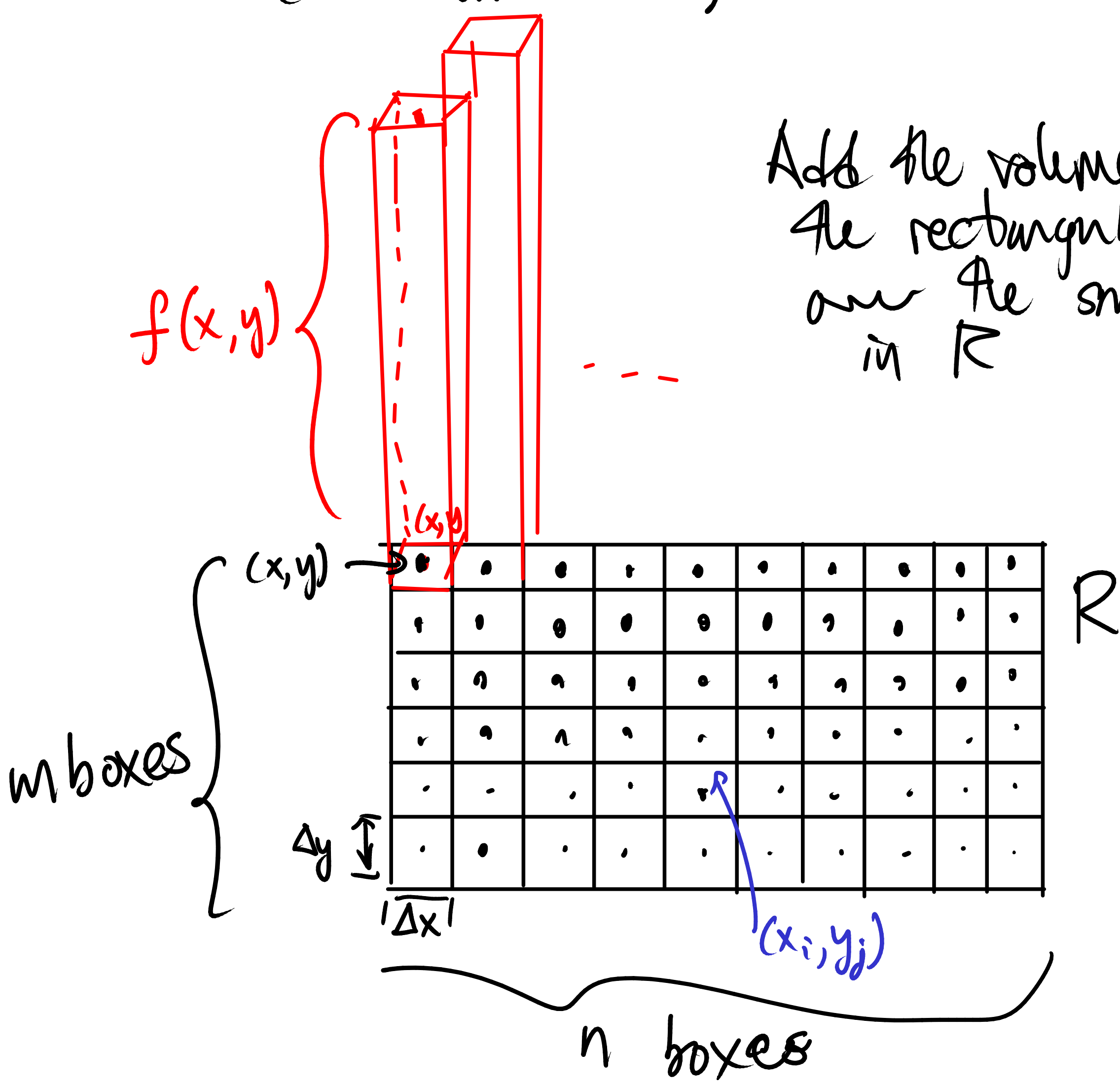
← subtract these. . . .



Fill with rectangular prisms

$V = L \cdot W \cdot H$

Realize we need to do the systematically.
(Riemann Sum)



Add the volumes of the rectangular prisms over the small rectangles in R

$$\text{Volume} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\Delta x \Delta y = \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta A$$

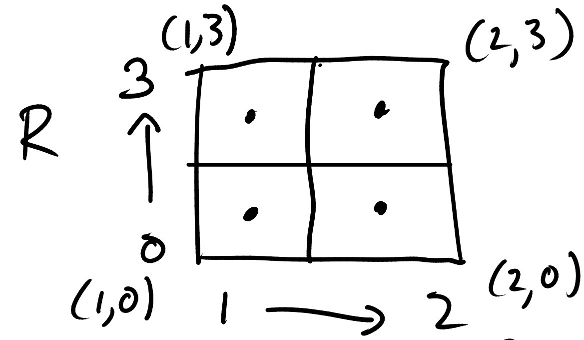
where $\Delta A = \Delta x \Delta y$ is the area of one of the boxes in R

There's a lot of freedom in how you set up the Riemann sum.

If f is nice (continuous is sufficient) then as $n \to \infty$ and $m \to \infty$, and the boxes get smaller, all Riemann sums converge to the same value. Call this common limit the integral

$$V = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

<u>Example</u> $f(x,y) = 1 + x^2 + 3y$



Riemann sum with $n = m = 2$

$\Delta x = \frac{1}{2}$ $\quad \Delta y = \frac{3}{2}$

$\Delta A = \frac{3}{4}$

$$R = \{1 \le x \le 2,\ 0 \le y \le 3\} = \underset{x}{[1,2]} \times \underset{y}{[0,3]}$$

$f(\frac{5}{4}, \frac{9}{4})$	$f(\frac{7}{4}, \frac{9}{4})$
$f(\frac{5}{4}, \frac{3}{4})$	$f(\frac{7}{4}, \frac{3}{4})$

Average value $f(x,y)$: $f_{ave} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$

Properties: Linearity $\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$

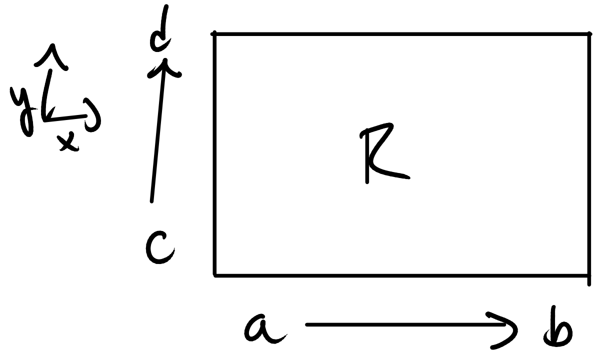
$$\iint_R cf\,dA = c\iint_R f\,dA \quad c \text{ a constant}$$

Monotonicity if $f(x,y) \geq g(x,y)$ for all (x,y) in R

then $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

Caveat: Integral is actually the signed volume
its negative if $f(x,y) < 0$

In practice, we use iterated integrals and FTC.



$\{a \leq x \leq b,\ c \leq y \leq d\}$

$[a,b] \times [c,d]$

$$V = \iint_R f\,dA$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$:= \int_c^d \left\{ \int_a^b f(x,y)\,dx \right\} dy$$

integrate W.R.T. x, then W.R.T. y

This is called an iterated integral.

$$\iint_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left\{ \int_c^d f(x,y)\,dy \right\} dx$$

Note: match integral sign $\int$ with differential $d\_$ from inside out.

<u>Examples</u> $\int_0^5 12x^2y^3\,dx$ (integrate w.r.t. x while holding y constant — integral analog of a partial derivative)

$$= \left[12\frac{x^3}{3} y^3 \right]_{x=0}^{x=5}$$

$$= 12\frac{5^3}{3}y^3 - 12\frac{0^3}{3}y^3 = \frac{12\cdot 5^3}{3}y^3$$

The result is a function of y, but not of x!

<u>Example</u> $$\int_1^3 \int_1^5 \frac{\ln y}{xy}\,dy\,dx = \int_1^3 \left\{ \int_1^5 \frac{\ln y}{xy}\,dy \right\} dx$$

$$\text{inner} = \int_1^5 \frac{\ln y}{xy}\,dy \qquad u = \ln y \quad du = \frac{1}{y}dy$$

$y = 1 \to u = \ln 1 = 0$

$y = 5 \to u = \ln 5$

$$= \int_0^{\ln 5} \frac{u\,du}{x}$$

$$= \left[\frac{1}{x}\frac{u^2}{2} \right]_{u=0}^{u=\ln 5} = \frac{1}{x}\frac{(\ln 5)^2}{2} - 0 = \frac{(\ln 5)^2}{2}\frac{1}{x}$$

outer $= \int_1^3 \frac{(\ln 5)^2}{2}\frac{1}{x}\,dx = \frac{(\ln 5)^2}{2}\int_1^3 \frac{1}{x}\,dx = \boxed{\frac{(\ln 5)^2}{2}\ln 3}$

$$\iint_R \sin(x-y)\,dA \qquad R=\left\{0\le x\le \frac{\pi}{2},\ 0\le y\le\frac{\pi}{2}\right\}$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\sin(x-y)\,dx\,dy \quad \text{or} \quad \int_0^{\pi/2}\int_0^{\pi/2}\sin(x-y)\,dy\,dx$$

x first $\qquad\qquad$ y first

$$=\int_0^{\pi/2}\left\{\int_0^{\pi/2}\sin(x-y)\,dx\right\}dy$$

$$=\int_0^{\pi/2}\left\{\Big[-\cos(x-y)\Big]_{x=0}^{x=\frac{\pi}{2}}\right\}dy$$

$$=\int_0^{\pi/2}\left\{-\cos\left(\frac{\pi}{2}-y\right)+\cos(0-y)\right\}dy$$

just a single variable integral ...

Ex: $\displaystyle\iint_R \frac{xy^2}{x^2+1}\,dA \qquad R=\{0\le x\le 1,\ -3\le y\le 3\}$

$$=\int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx \qquad \frac{xy^2}{x^2+1}=\left(\frac{x}{x^2+1}\right)\left(y^2\right)$$

$$= \int_0^1 \left\{ \frac{x}{x^2+1} \int_{-3}^{3} y^2\,dy \right\} dx$$

$$= \left(\int_0^1 \frac{x}{x^2+1}\,dx \right) \left(\int_{-3}^{3} y^2\,dy \right)$$

General fact $\displaystyle\int_c^d \int_a^b g(x)\cdot h(y)\,dx\,dy$

$$= \left(\int_a^b g(x)\,dx \right) \left(\int_c^d h(y)\,dy \right)$$ if you can factor the function

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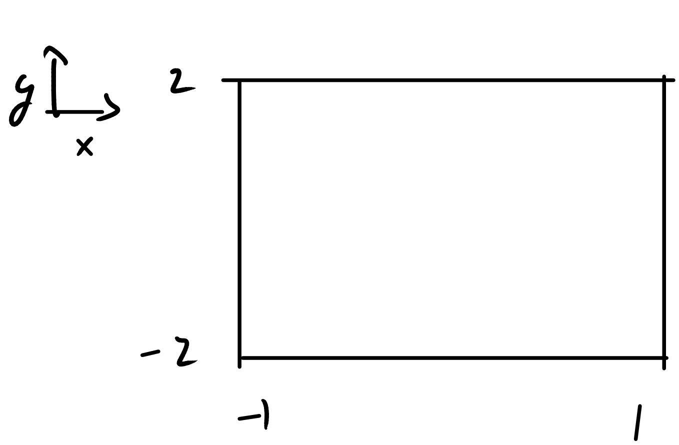
Volume under paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$

above rectangle $R = [-1,1] \times [-2,2]$



$f(x,y) = z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$

$-1 \le x \le 1 \Rightarrow x^2 \le 1$

$\frac{x^2}{4} \le \frac{1}{4}$

$-2 \le y \le 2 \Rightarrow y^2 \le 4$

$\frac{y^2}{9} \le \frac{4}{9} < \frac{1}{2}$

Indeed $f(x,y)$ is positive on this whole rectangle

Volume $= \iint_R f(x,y)\,dx\,dy$

$$V = \int_{-1}^{1} \int_{-2}^{2} \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dy\, dx$$