Lagrange Multipliers: Constrained Optimization

Problem: Maximize (or minimize) $f(x,y,z)$
subject to constraint $g(x,y,z) = C$
(C = constant)

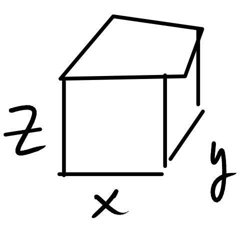
$P = bL^{\alpha}K^{1-\alpha}$ Cobb-Douglas production func.
$0 < \alpha < 1$

L = labor
K = capital
P = production

C = cost
$C = mL + nK$

Problem: Maximize P for a fixed cost C
Minimize C for a fixed production P.

Rectangular:



Maximize Volume $= xyz$
for a fixed surface area

$A = 2xy + 2xz + 2yz$

One way to solve: solve constraint equation and substitute into f

Eg. minimize $f(x,y) = x^2+y^2$ subject to $x+y=1$

constraint equation $x+y=1 \Rightarrow y = 1-x$

$f(x,y) = f(x,1-x) = x^2+(1-x)^2 = x^2+1-2x+x^2$

$f = 2x^2 - 2x + 1 \quad 0 = \frac{d}{dx}(2x^2 - 2x + 1) = 4x - 2$

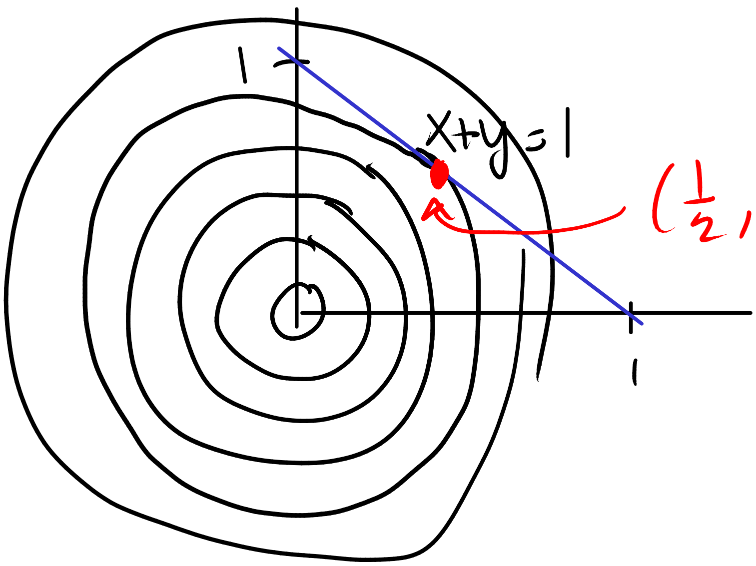
$\Downarrow$

$x = \frac{1}{2}$

$y = 1 - x = \frac{1}{2}$

$f(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$f = (\text{dist to } 0)^2$



Observe: constraint curve is tangent to the level curve of f at the minimum.

Not always possible to solve constraint equation.
Lagrange multipliers let us optimize without solving constraint.

<u>2D Lagrange multipliers</u> Optimize $f(x,y)$ subject to $g(x,y) = c$

Solve
$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = c \end{cases}$$

λ is a new variable called the Lagrange multiplier

3 equations $\left\{\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = c \end{array}\right\}$ 3 variables x, y, λ

$\underline{\text{Ex}}$ $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x + y = 1$

$$\nabla f = \lambda \nabla g \qquad \nabla f = \langle 2x, 2y \rangle$$
$$g(x,y) = 1 \qquad \nabla g = \langle 1, 1 \rangle$$

$$\langle 2x, 2y \rangle = \lambda \langle 1, 1 \rangle = \langle \lambda, \lambda \rangle$$

$\left.\begin{array}{l} 2x = \lambda \\ 2y = \lambda \\ x + y = 1 \end{array}\right\}$ $x = \frac{\lambda}{2}$ $y = \frac{\lambda}{2}$

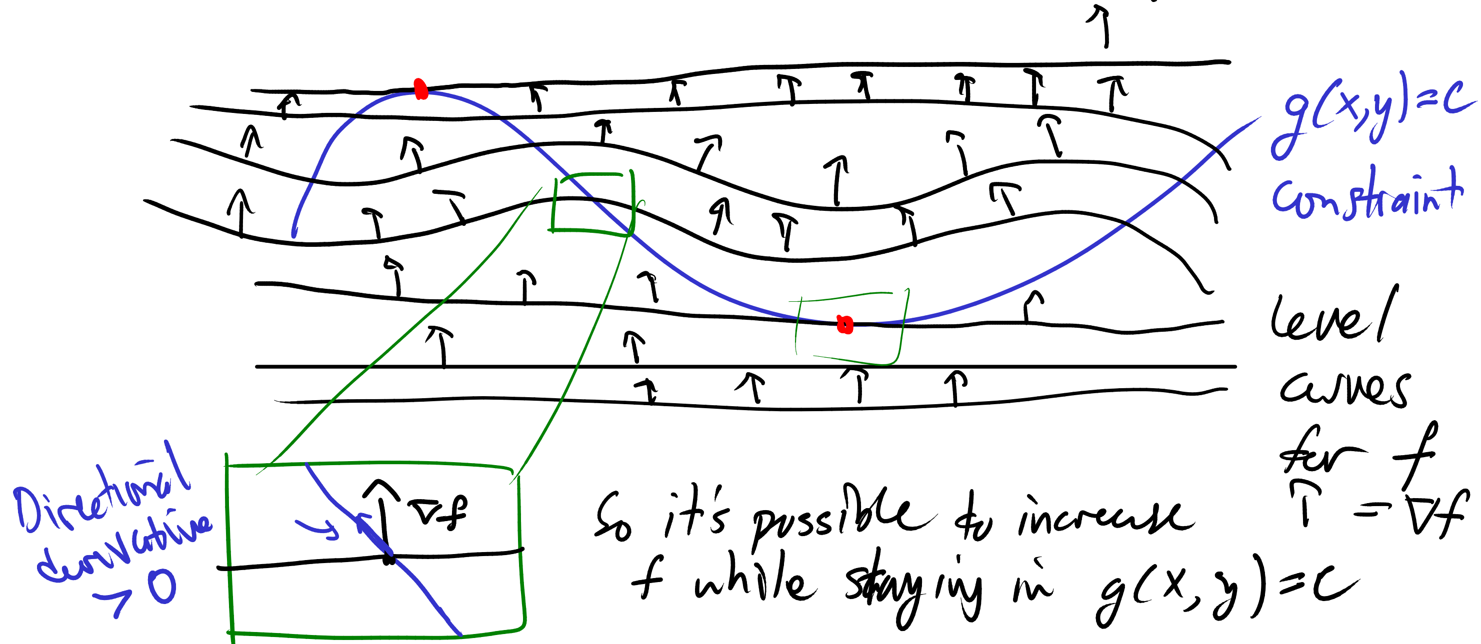
$\frac{\lambda}{2} + \frac{\lambda}{2} = 1$ $\lambda = 1$ $x = \frac{1}{2}$ $y = \frac{1}{2}$

Minimum occurs at $\left(\frac{1}{2}, \frac{1}{2}\right)$

minimum value $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$

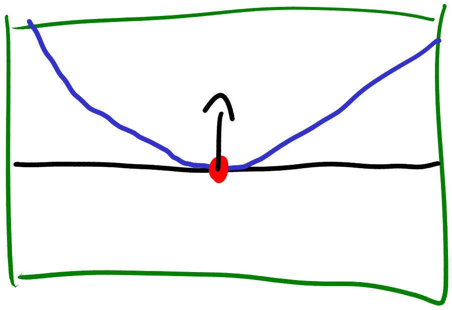
Multiplier equation says ∇f is proportional to ∇g

$$\nabla f = \lambda \nabla g$$

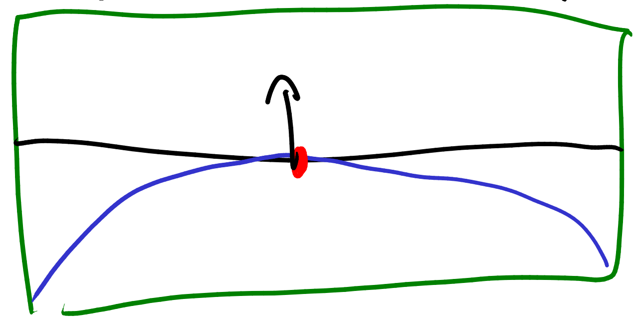


So it's possible to increase f while staying in $g(x,y) = c$

constrained minimum.



constrained maximum



To summarize: at a constrained max or min

* The level curve for f is tangent to the curve $g(x,y)=c$.

∇f perp. to level curve of f

$\Leftrightarrow$ ∇f perp to $g(x,y)=c$ (because level curve of f tangent to $g(x,y)=c$)

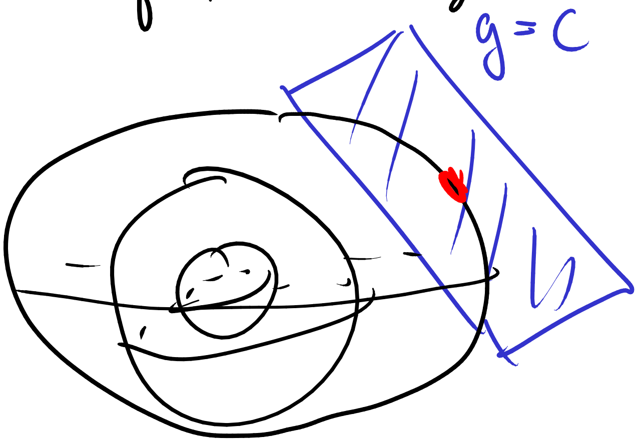
OTOH ∇g is perp to $\underline{g(x,y)=c}$
this is a level curve of g

∇f and ∇g are perpendicular to same thing, so they are parallel: $\boxed{\nabla f = \lambda \nabla g}$

3D Lagrange multipliers: Optimize $f(x,y,z)$
subject to $g(x,y,z)=c$
(constraint surface)

Solve $\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z)=c \end{cases}$

(maxima & minima occur when level surface of f is tangent to constraint surface $g=C$)



- 4 equations in 4 unknowns x, y, z, λ

Ex $f(x,y,z) = x^2+y^2+z^2$

Constraint $g(x,y,z) = x^4+y^4+z^4 = 1$

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 4x^3, 4y^3, 4z^3 \rangle$

$\nabla f = \lambda \nabla g$: $\quad 2x = \lambda 4x^3$ $\quad x^4+y^4+z^4 = 1$

$2y = \lambda 4y^3$

$2z = \lambda 4z^3$

$2x = \lambda 4x^3$

$x = \lambda 2x^3 \Rightarrow x = 0$ or $1 = \lambda \cdot 2x^2$

$y = \lambda 2y^3 \Rightarrow y = 0$ or $1 = \lambda \cdot 2y^2$

$z = \lambda 2z^3 \Rightarrow z = 0$ or $1 = \lambda \cdot 2z^2$

Suppose x, y, z are all not zero.

$x^2 = \frac{1}{2\lambda} \quad y^2 = \frac{1}{2\lambda} \quad z^2 = \frac{1}{2\lambda}$

$$1 = x^4 + y^4 + z^4 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2$$

$$1 = 3\frac{1}{4\lambda^2} \qquad \frac{4\lambda^2}{3} = 1 \qquad \lambda^2 = \frac{3}{4}$$

$$\lambda = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

$x^2 = \frac{1}{2\lambda} = \pm\frac{1}{\sqrt{3}}$, minus is impossible

$$x^2 = \frac{1}{\sqrt{3}} \qquad x = \pm\frac{1}{\sqrt[4]{3}}$$

So get solutions $\lambda = \frac{1}{\sqrt{3}}$, $(x,y,z) = \left(\pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[3]{3}}, \pm\frac{1}{\sqrt[4]{3}}\right)$

8 solutions already!

Also need to consider what happens when

$x = 0 \quad y, z \neq 0$

$x = 0, y = 0, z \neq 0$ etc.

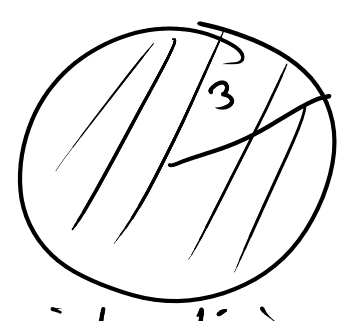
<u>Example</u> Global max/min for

$f(x,y) = x^2 + y^2 + 4x - 4y$ on disk $x^2 + y^2 \leq 9$

(i) critical points in interior

(ii) max/min on boundary

$$x^2 + y^2 = 9$$



Can think of (ii) as a constrained optimization

(i) $\nabla f = \langle 2x+4, 2y-4 \rangle = 0$

$2x+4=0 \qquad x=-2 \qquad (-2,2)$

$2y-4=0 \qquad y=2$

$x^2+y^2 = (-2)^2+(2)^2 = 4+4 = 8 \leq 9$

So this critical point lives inside disk $x^2+y^2 \leq 9$

$f(-2,2) = (-2)^2+(2)^2+4(-2)-4(2) = -8$

(ii) Max/min of f on curve $g(x,y) = x^2+y^2 = 9$

$\nabla f = \lambda \nabla g \qquad \langle 2x+4, 2y-4 \rangle = \lambda \langle 2x, 2y \rangle$

$2x+4 = \lambda \cdot 2x \rightarrow (2-2\lambda)x+4=0$

$2y-4 = \lambda 2y \qquad x = \frac{-4}{2-2\lambda} = \frac{2}{\lambda-1}$

$\downarrow$

$(2-2\lambda)y = 4 \rightarrow \quad y = \frac{4}{2-2\lambda} = \frac{2}{1-\lambda}$

$\left(\frac{2}{\lambda-1}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 = 9$

$\frac{4}{(\lambda-1)^2} + \frac{4}{(\lambda-1)^2} = 9 \qquad \frac{8}{(\lambda-1)^2} = 9$

$(\lambda-1)^2 = \frac{8}{9} \qquad (\lambda-1) = \pm \frac{2\sqrt{2}}{3}$

$\lambda = 1 \pm \frac{2\sqrt{2}}{3}$

$$x = \frac{2}{\lambda - 1} = \frac{2}{\pm \frac{2\sqrt{2}}{3}} = \pm \frac{3}{\sqrt{2}}$$

$$y = \frac{-2}{\lambda - 1} = -x$$

$$\left(x = \frac{3}{\sqrt{2}},\ y = -\frac{3}{\sqrt{2}}\right), \quad f = \left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{-3}{\sqrt{2}}\right)^2 + 4\left(\frac{3}{\sqrt{2}}\right) - 4\left(\frac{-3}{\sqrt{2}}\right)$$

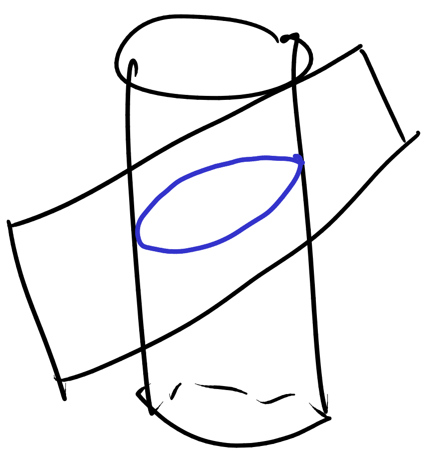
$$= \frac{9}{2} + \frac{9}{2} + \frac{8 \cdot 3}{\sqrt{2}}$$

$$\left(x = \frac{-3}{\sqrt{2}},\ y = \frac{3}{\sqrt{2}}\right)$$

$$f = \left(\frac{-3}{\sqrt{2}}\right)^2 + \left(\frac{3}{\sqrt{2}}\right) + 4\left(\frac{-3}{\sqrt{2}}\right) - 4\left(\frac{3}{\sqrt{2}}\right)$$

$$= \frac{9}{2} + \frac{9}{2} - \frac{8 \cdot 3}{\sqrt{2}}$$

<u>Multiple constraints</u>: Optimize $f(x,y,z)$ subject to two constraints $g(x,y,z) = c$, $h(x,y,z) = k$



Two Lagrange multipliers λ, μ

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$g(x,y,z) = c, \quad h(x,y,z) = k$$

5 equations in 5 variables x, y, z, λ, μ.