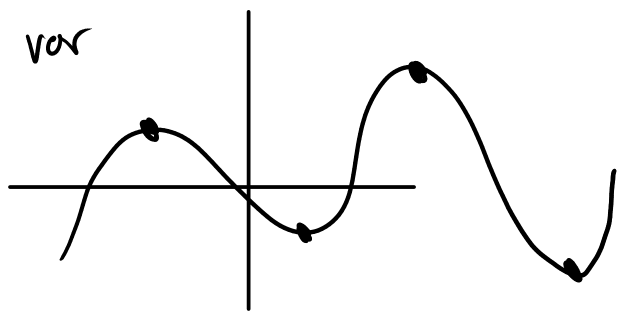
Maximum and minimum values.

Basic question: fix maximum and minimum of a function $f(x,y)$ of 2 variables

1 var



local maximum (x_0,y_0)

$f(x_0,y_0) \geq f(x,y)$ for all points (x,y) close to (x_0,y_0)

global maximum : $f(x_0,y_0) \geq f(x,y)$ for all (x,y) in the domain of f

domain of f = "feasible set"

· local minimum and global minimum are analogous

Necessary condition for max or min: <u>derivatives are zero</u> (if they exist)

In 1 var: $f(x)$ has local max/min at $a \Rightarrow f'(a)=0$

In 2 var: $f(x,y)$ has a local max/min at (a,b)

$$\Rightarrow f_x(a,b)=0 \text{ and } f_y(a,b)=0$$

same as : $\nabla f(a,b) = 0$

One way to see this: Suppose $f(x,y)$ local max at (a,b)

Consider freeze $x=a$: $g(y)=f(a,y)$

$g(y)$ has a local max at $y=b$: $g'(b)=0$

$$0=g'(b)=f_y(a,b)$$

Freeze $y=b$: $h(x)=f(x,b)$

$h(x)$ has local max at a $h'(a)=0$

$$0=h'(a)=f_x(a,b)$$

Same argument for local minimum.

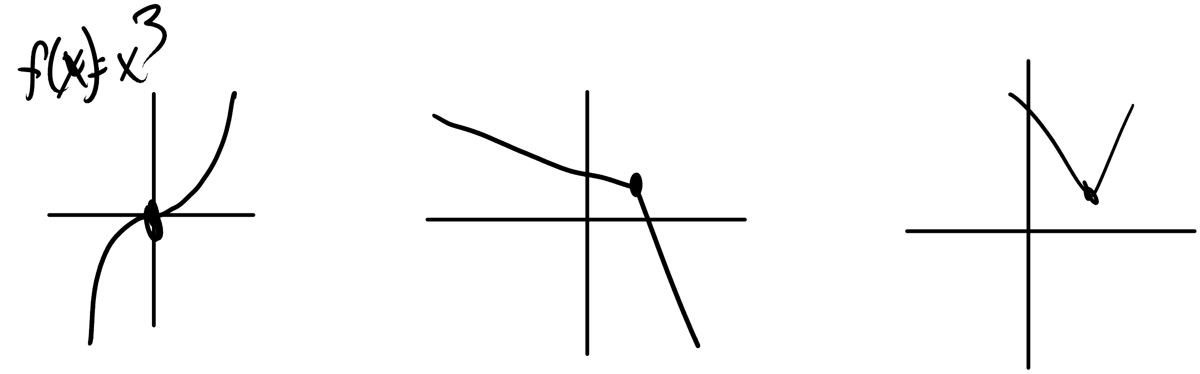
Thm: If $f(x,y)$ has a local max or min at (a,b) and the partial derivatives f_x and f_y exist then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Def: (a,b) is a Critical point of $f(x,y)$ if $f_x(a,b)=0$ and $f_y(a,b)=0$, or if one of the derivatives does not exist.

* Any local max or min is a critical point.

A critical point is not necessarily a max or min.

Eg.



2 var: $f(x,y) = x^2 - y^2$ $\quad f_x = 2x$
$\quad f_y = -2y$

$f_x(0,0)=0 \quad f_y(0,0)=0 \quad (0,0)$ is critical point.

But $(0,0)$ is neither a max or a min
It is a saddle.

In the graph of a function of two variables critical points are points where the tangent plane is horizontal.

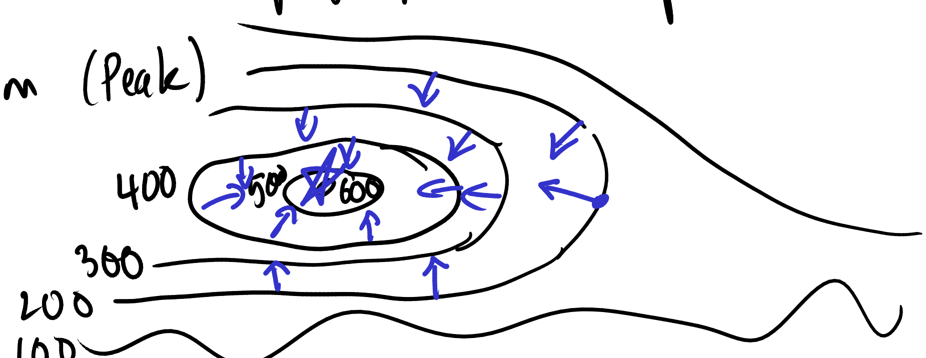
$$z - z_0 = \cancel{f_x(x_0,y_0)}(x-x_0) + \cancel{f_y(x_0,y_0)}(y-y_0)$$

$$z = z_0$$

Contour plots: (Topographical map z= elevation)

Maximum (Peak)

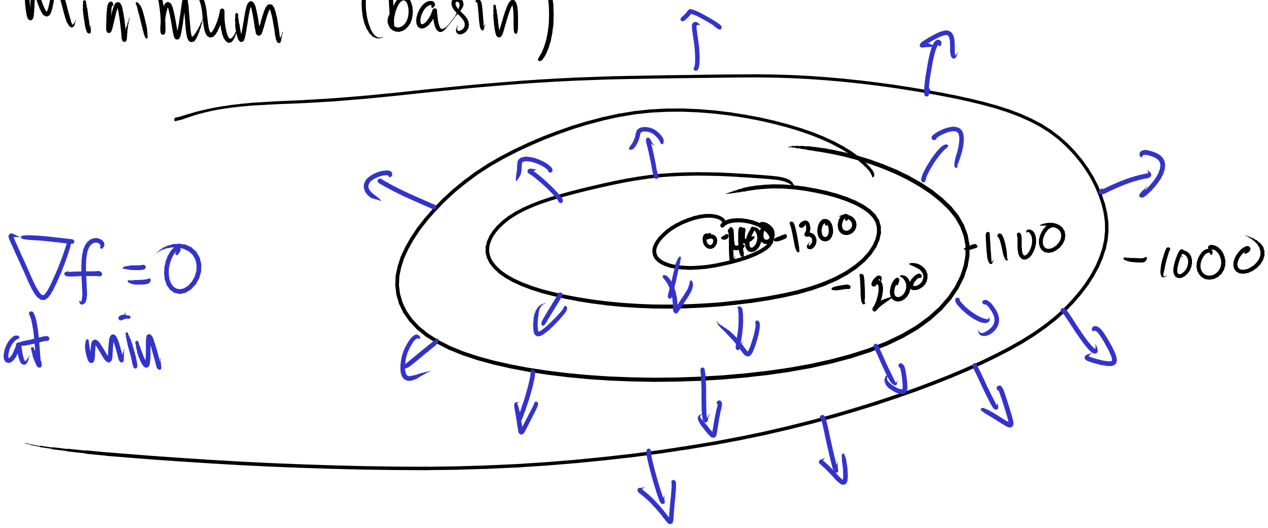
$\nabla f = \langle f_x, f_y \rangle$



points uphill perpendicular to contours.

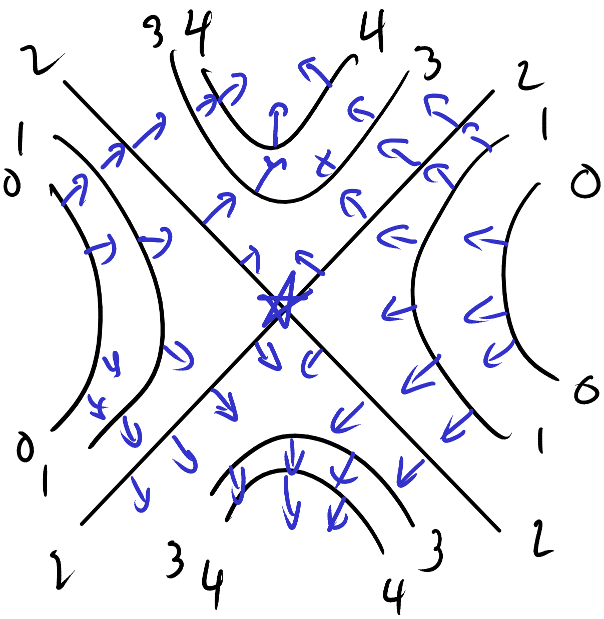
At max $\nabla f = 0$.

Minimum (basin)



$\nabla f = 0$ at min

Saddle



$\nabla f = 0$ at saddle (where lines cross)

Mountain pass / gap.

How to tell which is which: second derivative test.

1 var: if $f'(a) = 0$ and $f''(a) > 0 \Rightarrow a$ is local min.
if $f'(a) = 0$ and $f''(a) < 0 \Rightarrow a$ is local max
if $f'(a) = 0$ and $f''(a) = 0$
inconclusive (degenerate critical pt)

2 var: Define $D = f_{xx} f_{yy} - (f_{xy})^2$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} \quad \text{(Hessian Determinant)}$$

Assume (a,b) is a critical point: $f_x(a,b)=0$, $f_y(a,b)=0$.

(i) if $D>0$ and $f_{xx}>0$ then (a,b) is a local min
" " " $f_{yy}>0$ " " " " "

(ii) if $D>0$ and $f_{xx}<0$ then (a,b) is a local max
" " " $f_{yy}<0$ " " " " "

(iii) if $D<0$ then (a,b) is a saddle.

(iv) if $D=0$ the critical point is degenerate
(can't tell without more work)

<u>Example</u>: $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$

Find and classify the critical points.

$f_x = 6xy - 12x = 0 \Rightarrow x(6y-12)=0$

$\Downarrow$

$f_y = 3y^2 + 3x^2 - 12y = 0$ $\qquad x=0$ or $6y-12=0$

$x=0$ or $y=2$

If $x=0$: $f_y = 3y^2 - 12y = 0$

$y^2 - 4y = 0 \Rightarrow y=0$ or $y=4$

If $y=2$: $f_y = 3\cdot 4 + 3x^2 - 12\cdot 2 = 0$

$12 + 3x^2 - 24 = 0 \Rightarrow 3x^2 = 12$

$x^2 = 4 \Rightarrow x = 2$ or $x = -2$

4 critical points $(0,0)$, $(0,4)$

$(2,2)$, $(-2,2)$

$f_{xx} = 6y - 12$

$f_{yy} = 6y - 12$

$f_{xy} = 6x$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (6y-12)^2 - (6x)^2$

At $(0,0)$ $D = (-12)^2 = 144 > 0$ $f_{xx} = -12 < 0$ Max

At $(0,4)$ $D = (24-12) = 144 > 0$ $f_{xx} = 24 - 12 = 12 > 0$ Min

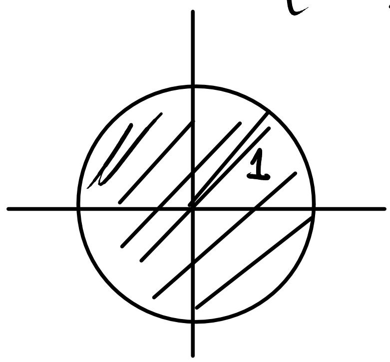
At $(2,2)$ $D = (12-12)^2 - (6\cdot 2)^2 = -144 < 0$ Saddle

At $(-2,2)$ $D = (12-12)^2 - (-12)^2 = -144 < 0$ saddle.

To Talk about global maxima and minima, need to describe the domain or feasible set.

Maximize $f(x,y) = x^2+y^2$

on the set $\{(x,y) \mid x^2+y^2 \le 1\}$



critical point $f_x = 2x = 0$
$f_y = 2y = 0$

critical point. $(0,0)$ min

Maximum occurs at the boundary $\{x^2+y^2=1\}$

To find global maxima and minima need to consider:

(i) critical points inside the domain

and (ii) maximum and minimum values on the boundary.

Application: Utility maximizations.

Can buy some amount of commodities 1, 2

x = amount of commodity 1 to buy
y = amount of " 2 to buy.

Prices $p_1 > 0$. $p_2 > 0$.

Amount spent $xp_1 + yp_2$

Amount of money available W

Maximize Utility $U(x,y)$

Find global maximum of $U(x,y)$

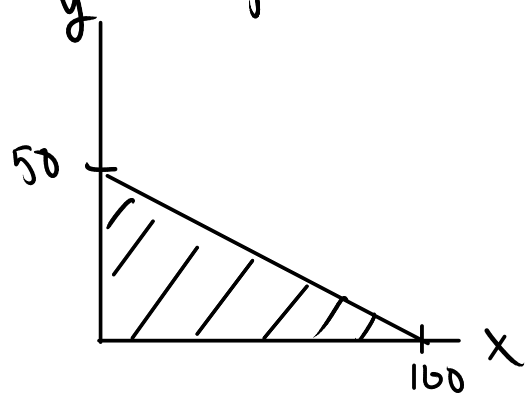
on the feasible set (domain) $x \geq 0$
$y \geq 0$
$xp_1 + yp_2 \leq W$

for definiteness

$p_1 = 1$ $\quad W = 100$

$p_2 = 2$ $\quad U(x,y) = x+y$

Feasible region $\quad x \geq 0 \quad y \geq 0$
$x + 2y \leq 100$



Consider

(1) critical points inside triangle.

$$\frac{\partial U}{\partial x} = 1 \qquad \frac{\partial U}{\partial y} = 1$$

No critical points!

(2) maximum along sides of the triangle

Side $y=0$ $\quad U(x,0) = x$

Maximum $x = 100$ $\quad U(100,0) = 100$

Side $x=0$ $\quad u(0,y)=y$ $\quad$ max $y=50$
$u=50$

Side $x+2y=100$ $\quad x=100-2y$

$u(100-2y, y)=100-y$ $\quad$ max at $y=0$
$x=100$
$u=100$

The corner $(x=100, y=0)$ is where the global max $u=100$ occurs.

Moral: spend all money on commodity 1.