Exam Thursday

Quadric surfaces

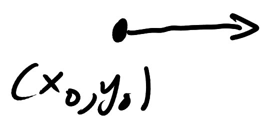
Chapter 13 vector functions

Chapter 14 through 14.5 (not including today)

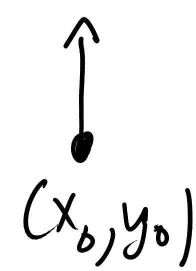
Today Directional Derivatives and Gradient Vectors

$f(x,y)$: $\frac{\partial f}{\partial x} = f_x$ = partial derivative w.r.t. x

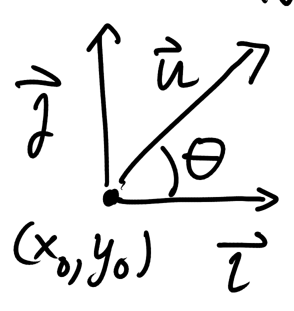
= derivative in the x-direction



$\frac{\partial f}{\partial y} = f_y$ = derivative in the y-direction



Can take derivatives in any direction!



$\vec{u} = \langle a,b \rangle = a\vec{i} + b\vec{j}$

is a unit vector $|\vec{u}| = 1$ $\sqrt{a^2+b^2} = 1$

$\vec{u} = \langle \cos\theta, \sin\theta \rangle$

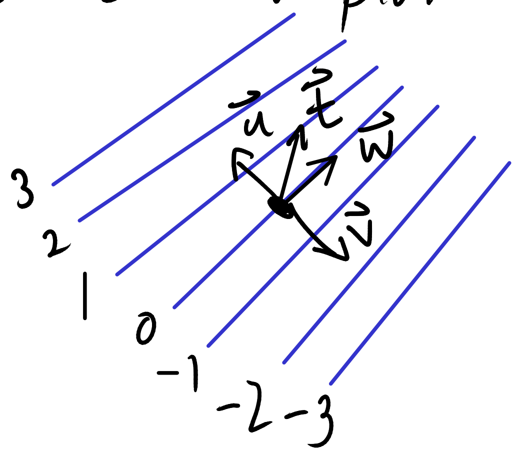
$D_{\vec{u}} f(x_0,y_0)$ = Directional Derivative at (x_0,y_0) in the direction $\vec{u}$.

value at distance h in the $\vec{u}$-direction

$$\lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h} =: D_{\vec{u}} f(x_0,y_0)$$

where $\vec{u} = \langle a,b \rangle$

Example contour plot.



$D_{\vec{u}} f > 0$ (increasing)

$D_{\vec{v}} f < 0$ (decreasing)

$D_{\vec{w}} f = 0$ (constant)

$$0 < D_{\vec{t}} f < D_{\vec{u}} f$$

We can compute all directional derivatives from knowing just $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$\vec{u} = \langle a,b \rangle$

$$\frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$$

$$= \frac{g(h) - g(0)}{h} \quad \text{where} \quad g(h) = f(x_0+ah, y_0+bh)$$

$$D_{\vec{u}} f(x_0,y_0) = g'(0)$$

$$g'(h) = \frac{d}{dh} g(h) = \frac{d}{dh} f(\overbrace{x_0+ah}^{x}, \overbrace{y_0+bh}^{y})$$

$$= \frac{\partial f}{\partial x}(x_0+ah, y_0+bh)\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}(x_0+ah, y_0+bh)\frac{\partial y}{\partial h}$$

$$= \frac{\partial f}{\partial x}(x_0+ah, y_0+bh)\cdot a + \frac{\partial f}{\partial y}(x_0+ah, y_0+bh)\, b$$

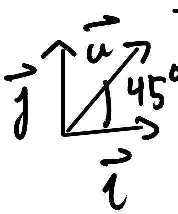
at $h=0$: $\frac{\partial f}{\partial x}(x_0,y_0)\cdot a + \frac{\partial f}{\partial y}(x_0,y_0)\cdot b = D_{\vec{u}} f(x_0,y_0)$

$\vec{u} = \langle a, b\rangle$

If $\vec{u} = \langle \cos\theta, \sin\theta\rangle$ is a unit vector:

$$D_{\vec{u}} f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\cos\theta + \frac{\partial f}{\partial y}(x_0,y_0)\sin\theta$$

<u>Ex</u> $f(x,y) = e^x \cos y \quad (x_0,y_0) = (0,0) \quad \theta = \frac{\pi}{4}$



$\vec{u}$ = unit vector at angle $\theta = \frac{\pi}{4}$ $\quad \vec{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$

$D_{\vec{u}} f(0,0)$: $\quad \frac{\partial f}{\partial x} = e^x \cos y \quad \frac{\partial f}{\partial x}(0,0) = 1\cdot 1 = 1$

$\quad \frac{\partial f}{\partial y} = -e^x \sin y \quad \frac{\partial f}{\partial y}(0,0) = 0$

$$D_{\vec{u}} f(0,0) = \frac{\partial f}{\partial x}\frac{\sqrt{2}}{2} + \frac{\partial f}{\partial y}\frac{\sqrt{2}}{2} = 1\cdot\frac{\sqrt{2}}{2} + 0\,\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

$\vec{u} = \langle a, b \rangle$

We have $D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$$

Define $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

called the Gradient vector of f.

$\nabla f(x,y)$ is a vector-valued function of x and y.

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

The gradient ∇f contains information about <u>all</u> directional derivatives.

<u>Examples</u>: $f(x,y) = y^2/x \quad (1,2) \quad \vec{u} = \frac{1}{3}(2\vec{i} + \sqrt{5}\vec{j})$

$D_{\vec{u}} f(1,2) = ?$ $\quad f_x(x,y) = -\dfrac{y^2}{x^2} \quad f_x(1,2) = -4$

$f_y(x,y) = \dfrac{2y}{x} \quad f_y(1,2) = 4$

$$\nabla f(x,y) = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x} \right\rangle = -\frac{y^2}{x^2}\vec{i} + \frac{2y}{x}\vec{j}$$

$$\nabla f(1,2) = \langle -4, 4 \rangle = -4\vec{i} + 4\vec{j}$$

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = \langle -4, 4 \rangle \cdot \frac{1}{3}\langle 2, \sqrt{5} \rangle$$

$$= \frac{1}{3}(-8 + 4\sqrt{5}).$$

Example $f(x,y) = \dfrac{x}{x^2+y^2}$ $(1,2)$ $\vec{v} = \langle 3,5 \rangle$

Take directional derivative of f in the direction of $\vec{v}$.

$f_x(x,y) = \dfrac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2}$ $f_x(1,2) = \dfrac{5-2}{25} = \dfrac{3}{25}$

$f_y(x,y) = \dfrac{-x}{(x^2+y^2)^2}(2y)$ $f_y(1,2) = \dfrac{-4}{25}$

$\vec{v}$ is <u>not</u> a unit vector! $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\langle 3,5 \rangle}{\sqrt{3^2+5^2}} = \dfrac{\langle 3,5 \rangle}{\sqrt{34}}$

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = \nabla f(1,2) \cdot \left(\frac{\vec{v}}{|\vec{v}|} \right)$$

$$= \left\langle \frac{3}{25}, \frac{-4}{25} \right\rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

$$= \frac{1}{25\sqrt{34}} (-11)$$

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3 variables: $\vec{x} = \langle x,y,z \rangle$ $\vec{x}_0 = \langle x_0, y_0, z_0 \rangle$

$\vec{u}$ unit vector $\vec{u} = \langle a,b,c \rangle$ $\sqrt{a^2+b^2+c^2} = 1$

$$D_{\vec{u}} f(x_0,y_0,z_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

$$D_{\vec u} f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\,a + f_y(x_0,y_0,z_0)\,b + f_z(x_0,y_0,z_0)\,c$$

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

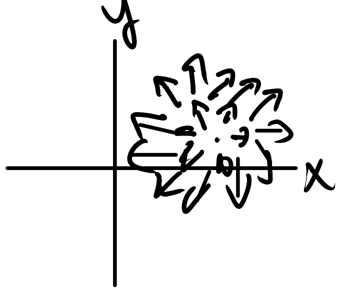
$$= \frac{\partial f}{\partial x}\vec i + \frac{\partial f}{\partial y}\vec j + \frac{\partial f}{\partial z}\vec k$$

$$D_{\vec u} f(x,y,z) = \nabla f(x,y,z) \cdot \vec u$$

Meaning of Gradient: Maximum Directional Derivative or Steepest Ascent.

$f(x,y)$ function

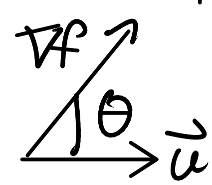
$\nabla f(x,y)$ gradient is a vector at every point



There should be a meaning for its magnitude and direction.

Right question: In what direction is f increasing the fastest.

$$D_{\vec u} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec u = |\nabla f(x_0,y_0)|\cdot|\vec u|\cos\theta$$



$$= |\nabla f(x_0,y_0)| \cos\theta$$

Greatest when $\theta = 0 \Rightarrow \cos\theta = 1$

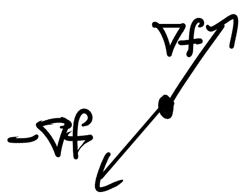
$$\Rightarrow \vec u \parallel \nabla f \quad \text{or} \quad \vec u = \frac{\nabla f}{|\nabla f|}$$

in this direction $D_{\left(\vec{u}=\frac{\nabla f}{|\nabla f|}\right)} f(x_0, y_0) = |\nabla f(x_0, y_0)|$

maximal value.

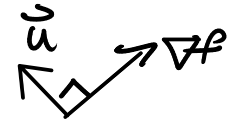
least when $\theta = \pi$ $\quad \cos\theta = -1$

$\vec{u} \parallel -\nabla f \qquad \vec{u} = -\dfrac{\nabla f}{|\nabla f|} \Rightarrow D_{\vec{u}} f(x_0, y_0) = -|\nabla f(x_0, y_0)|$

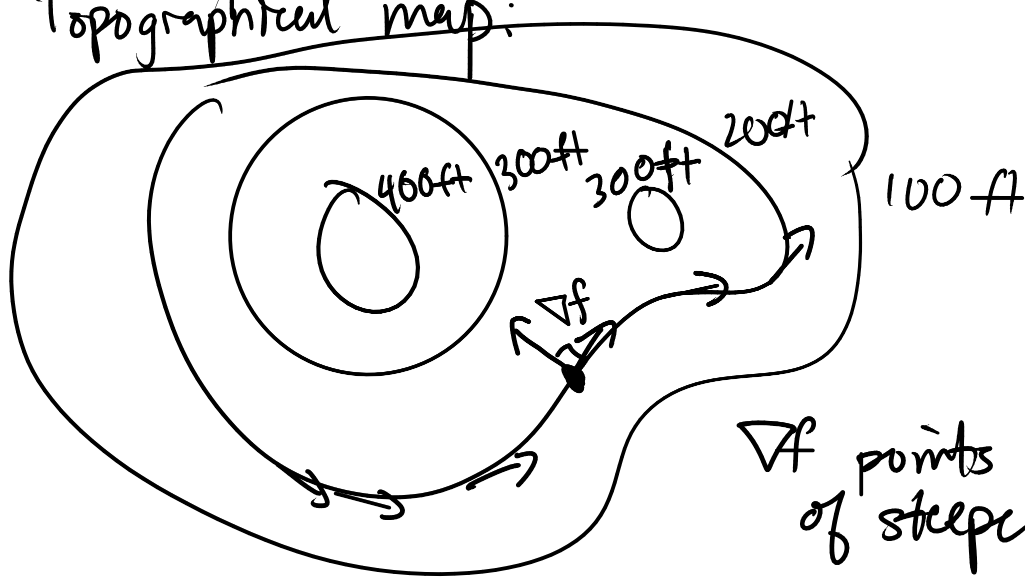


When is $D_{\vec{u}} f(x_0, y_0) = 0$? $\qquad \theta = \frac{\pi}{2} \Rightarrow \cos\theta = 0$

This happens when $\vec{u} \perp \nabla f$.



Topographical map:



f = elevation

∇f points in direction of steepest ascent

$-\nabla f$ points in direction of steepest descent.

Directions perp. to ∇f are where elevation doesn't change

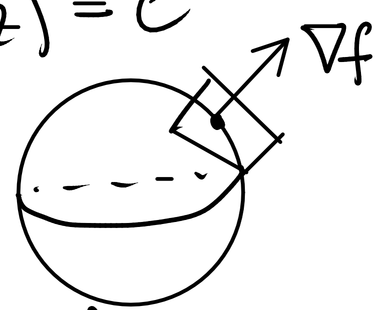
Directions $\vec{u}$ such that $\nabla f \cdot \vec{u} = 0$ are tangent to the level curves of $f(x,y)$

∇f is perpendicular to the level curve itself

In 3D: $f(x,y,z)$

level surface $f(x,y,z) = c$

$x^2+y^2+z^2 = C$



In this situation, $\nabla f(x,y,z)$ is perpendicular to the level surface.

Can find tangent plane to a level surface using ∇f as the normal vector

Tangent plane at (x_0,y_0,z_0) to $f(x_0,y_0,z_0) = c$.

$$\underbrace{\nabla f(x_0,y_0,z)}_{\vec{n}} \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$$f_x \cdot (x-x_0) + f_y \cdot (y-y_0) + f_z \cdot (z-z_0) = 0$$

$f(x,y,z) = x^2+y^2+z^2$ at $(1,1,1)$ on $f(x,y,z)=3$

$\nabla f = \langle 2x, 2y, 2z \rangle$ $\quad \nabla f(1,1,1) = \langle 2,2,2 \rangle$

Tangent plane: $\langle 2,2,2\rangle \cdot \langle x-1, y-1, z-1\rangle = 0$

$$2x + 2y + 2z = 6$$