Chain rule with multiple variables

Recall single-variable chain rule situation

- y is a function of x : $y = f(x)$
- x is a function of t : $x = g(t)$
- Thus, y is a function of t: $y = f(g(t))$

"y depends on t indirectly through x"

<u>Chain rule</u>: $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = f'(g(t)) \cdot g'(t)$

When we have more than one variable, the relationships of how things depend on each other can be more complicated.

At first we consider the situation

- z is function of x and y : $z = f(x,y)$
- x and y are each functions of t : $x = g(t)$, $y = h(t)$
- Thus z is a function of t through x and y: $z = f(g(t), h(t))$. We want to compute $\dfrac{dz}{dt}$.

We have

$$\boxed{\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= f_x(g(t), h(t))\, g'(t) + f_y(g(t), h(t))\, h'(t)\end{aligned}}$$

The formula involves partial derivatives of f, times the derivatives of x and y with respect to t.

To prove this, use linear approximation to $z = f(x,y)$

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

Divide by Δt:

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t}$$

Take limit as $\Delta t \to 0$: then

- $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$, $\frac{\Delta y}{\Delta t} \to \frac{dy}{dt}$, $\frac{\Delta z}{\Delta t} \to \frac{dz}{dt}$
- The two sides of the approximate equality get closer to each other.

So in the limit $\Delta t \to 0$, $\displaystyle \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Mnemonic: the chain rule is similar to the formula for a differential: $\displaystyle dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

<u>Caution</u>: The "cancelling differentials" mnemonic doesn't really work for the multivariable chain rule

Single variable: $\displaystyle \frac{dy}{\cancel{dx}}\frac{\cancel{dx}}{dt} = \frac{dy}{dt}$ Well, okay.

Multivariable:

$$\frac{\partial z}{\cancel{\partial x}}\frac{\cancel{dx}}{dt} + \frac{\partial z}{\cancel{\partial y}}\frac{\cancel{dy}}{dt} = \frac{dz}{dt} + \frac{dz}{dt} = 2\frac{dz}{dt} ???$$

WRONG

Examples: $z = x^2 + y^2 + xy$, $x = \sin t$, $y = e^t$

$$\frac{\partial z}{\partial x} = 2x + y \qquad \frac{\partial z}{\partial y} = 2y + x \qquad \frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = e^t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x+y)\cos t + (2y+x)e^t$$

$$= (2\sin t + e^t)\cos t + (2e^t + \sin t)e^t$$

$$= 2\sin t\cos t + 2(e^t)^2 + e^t\cos t + e^t \sin t$$

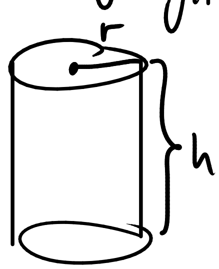
We can check this against substituting in for x and y

$$z = x^2 + y^2 + xy = (\sin t)^2 + (e^t)^2 + \sin t \cdot e^t$$

matches

$$\frac{dz}{dt} = 2\sin t\cos t + 2e^t \cdot e^t + \cos t \cdot e^t + \sin t \cdot e^t$$

Volume of cylinder $V = \pi r^2 h$



r increases at rate $3\,cm/s$

h decreases at rate $1\,cm/s$

What is $\frac{dV}{dt}$ when $r = 10cm$, $h = 20cm$?

chain rule: $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$ all evaluated at this particular moment.

$$\frac{\partial V}{\partial r} = 2\pi r h = 2\pi(10)(20) = 400\pi \qquad \frac{dr}{dt} = 3$$

$$\frac{\partial V}{\partial h} = 2\pi r^2 = 2\pi(10)^2 = 200\pi \qquad \frac{dh}{dt} = -1$$

$$\frac{dV}{dt} = (400\pi)3 + (200\pi)(-1) = 1000\pi \approx 3141 \text{ cm}^3/s$$

Another situation: $z = f(x,y)$, but each of x and y are functions of two more variables s and t.

$$\begin{cases} x = g(s,t) \\ y = h(s,t) \end{cases}$$

Then $z = f(g(s,t), h(s,t))$ is indirectly a function of the two variables s and t.

We can ask for $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

For $\frac{\partial z}{\partial s}$ it's like a usual derivative, but we hold t constant,

so $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$ by previous case

Similar for $\frac{\partial z}{\partial t}$: s is held constant:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

<u>Example</u>: $z = x^2y^3$, $x = s\cos t$, $y = s\sin t$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $(s,t) = (1, \frac{\pi}{4})$

Note: $(s,t) = (1, \frac{\pi}{4}) \Rightarrow x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}} \Rightarrow z = \frac{1}{4\sqrt{2}}$

$$\frac{\partial z}{\partial x} = 2xy^3 = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{2}{4} = \frac{1}{2}$$

$$\frac{\partial z}{\partial y} = 3x^2y^2 = 3\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{4}$$

$$\frac{\partial x}{\partial s} = \cos t = \frac{1}{\sqrt{2}} \qquad \frac{\partial y}{\partial s} = \sin t = \frac{1}{\sqrt{2}}$$

$$\frac{\partial x}{\partial t} = -s\sin t = -\frac{1}{\sqrt{2}} \qquad \frac{\partial y}{\partial t} = s\cos t = \frac{1}{\sqrt{2}}$$

Thus $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{4\sqrt{2}}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{1}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{2}}$$

General case: z depends on some number of variables

$$z = f(x_1, x_2, x_3, \dots, x_n)$$

Each x_i depends on other variables $t_1, t_2, t_3, \dots, t_m$

$$x_i = g_i(t_1, t_2, t_3, \dots, t_m)$$

Then z depends on $t_1, t_2, \dots, t_m$ through $x_1, x_2, \dots, x_n$

And
$$\frac{\partial z}{\partial t_1} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$

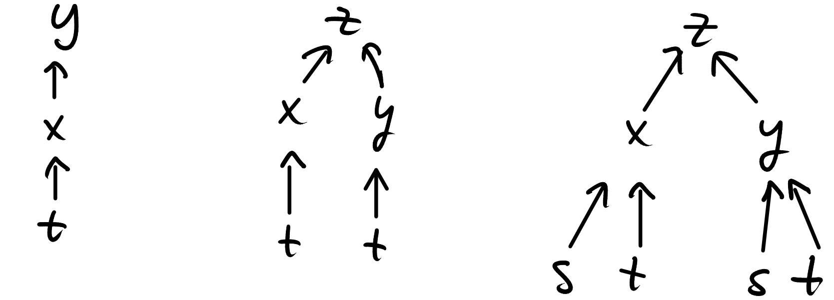
$$\frac{\partial z}{\partial t_2} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{\partial z}{\partial t_m} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

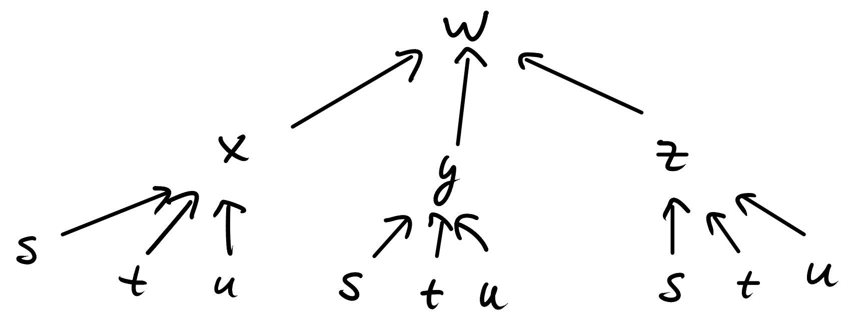
Tree diagrams can help organize the relationships

e.g.



arrow indicates "flow of data".

<u>Example</u>: $w = xe^{yz}$, $x = s^2tu$, $y = st^2u$, $z = stu^2$



<u>For example</u>: $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$

Implicit differentiation:

Curve in 2d defined by $F(x,y) = 0$ (e.g. $x^2+y^2-1=0$) $\boxed{\text{y is implicit function of x}}$

Take derivative with respect to x:

$$\frac{d}{dx}\big(F(x,y)\big) = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$ since $F(x,y)$ is constant.

$\frac{dx}{dx} = 1$, solve for $\frac{dy}{dx}$: $$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

This shows us how to find slope by differentiating defining eq.

Surface in 3d: $F(x,y,z)=0$
$(\text{eg. } z^2+x^2-y^2=0)$

$\boxed{z \text{ is implicit functi[on] of } x \text{ and } y}$

Can find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using same trick

$$0=\frac{\partial}{\partial x}\Big(F(x,y,z)\Big)=\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1}+\frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

solve for this

$$\frac{\partial z}{\partial x}=\frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{F_x}{F_z}$$

similarly $\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=-\frac{F_y}{F_z}$