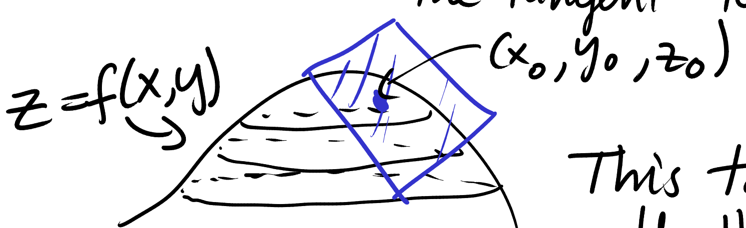
Tangent Planes and linear approximation

Recall in single variable:

- Graph $y = f(x)$
- slope of tangent line at (x_0, y_0): $f'(x_0)$
- equation of tangent line $y - y_0 = f'(x_0)(x - x_0)$
- linear approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

We will generalize this to functions of several variables, using partial derivatives.

<u>Tangent planes:</u> For a function of two variables, the tangent to the graph is a <u>plane</u>.



This tangent plane contains all the tangent lines of all the paths on the surface at the point $(x_0, y_0, z_0 = f(x_0, y_0))$

To derive equation for tangent plane:
General plane through (x_0, y_0, z_0):

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

divide by C and rearrange

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

Can determine a and b from partial derivatives

Hold y constant $y = y_0$

Tangent plane becomes $z - z_0 = a(x - x_0)$, $y = y_0$
tangent line to curve $\begin{cases} x = t \\ y = y_0 \\ z = f(t, y_0) \end{cases}$

We saw last time that the slope of this curve is $f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$

so $a = f_x(x_0, y_0)$

Similarly, holding x constant, $x = x_0$, we find

$$b = f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

Equation of tangent plane at $(x_0, y_0, z_0 = f(x_0, y_0))$

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Example $f(x, y) = 3y^2 - 2x^2 + x$ at $(2, -1, -3)$

$f_x = -4x + 1$ $\qquad f_x(2, -1) = -7$
$f_y = 6y$ $\qquad f_y(2, -1) = -6$

Tangent plane: $z - (-3) = (-7)(x - 2) + (-6)(y - (-1))$

Linear (or tangent plane) approximation

A major idea in calculus is that the tangent line may serve as an approximation to the function $f(x)$.

For a function of two variables, we use the tangent plane in the same way

Graph: $z = f(x,y)$

Tangent plane: $z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

If the function f is nice (differentiable is the term), then these two formulas for z are approximately equal, at least when x is close to x_0, and y is close to y_0.

$$\boxed{f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

Since x and y only appear to first power, this is called the <u>linear approximation</u>.

<u>Example</u>: $f(x,y) = x^3y^4$ at $(x_0,y_0) = (1,1)$

$f(1,1) = 1$

$f_x = 3x^2y^4 \qquad f_x(1,1) = 3$

$f_y = 4x^3y^3 \qquad f_y(1,1) = 4$

$$f(x,y) \approx 1 + 3(x-1) + 4(y-1)$$

Use this to compute $f(1.1, 0.9)$

$f(1.1, 0.9) \simeq 1 + 3(.1) + 4(-.1) = 1 + .3 - .4 = 0.9$

whereas $(1.1)^3(0.9)^4 = .873269$ exactly

The textbook and the homework use the notation $L(x,y)$ to denote the linearization

$$L(x,y) = f(x,y) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

<u>Technical point</u>: the linear approximation is only valid near (x_0,y_0) if f is <u>differentiable</u> at (x_0,y_0). If f_x and f_y are continuous at (x_0,y_0), then f is differentiable and the approx. is valid.

<u>Differentials</u> are a convenient notation for linear approx.

$z = f(x,y) \quad z_0 = f(x_0,y_0)$

$$z - z_0 \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

write $\Delta x = x - x_0$

$\Delta y = y - y_0$

$\Delta z = z - z_0 = f(x,y) - f(x_0,y_0)$

so $\Delta z \approx f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y$

is another way to write linear approx.

Introduce new formal symbols dx, dy, dz "differentials"

For independent variables: $dx = \Delta x$, $dy = \Delta y$.

But for dependent variable, dz is defined by linear approximation formula:

$$dz \overset{\text{definition}}{=} f_x(x,y)\Delta x + f_y(x,y)\Delta y$$
$$= f_x(x,y)\,dx + f_y(x,y)\,dy$$

OR : $$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

So $$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$$

and the approximation says $\boxed{\Delta z \approx dz.}$

<u>Example</u>: $f(x,y) = xe^y \quad f_x = e^y \quad f_y = xe^y$

$$dz = e^y dx + xe^y dy$$

We knew $f(1,0) = 1e^0 = 1$
Based on this, we approximate $f(1.1, 0.1)$

At $(x,y)=(1,0)$: $dz = e^0 dx + 1e^0 dy = dx + dy$

Now $dx = 1.1 - 1 = 0.1$ so $dz = 0.1 + 0.1 = 0.2$
$dy = 0.1 - 0 = 0.1$

Now $f(1.1, 0.1) - f(1,0) = \Delta z \approx dz = 0.2$

So $f(1.1,0.1) \approx f(1,0) + 0.2 = 1 + 0.2 = 1.2$

Computer approx: $f(1.1, 0.1) \approx 1.21569$, $\Delta z = .21569$

Three (or more variables) $w = f(x, y, z)$

Linear approximation at $(x, y, z) = (a, b, c)$

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

linearization $L(x,y,z) =$ Right-hand side of above $\uparrow$

Differential

$$dw = f_x\,dx + f_y\,dy + f_z\,dz = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$