Limits & partial derivatives:

<u>Def</u> $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$ means

For every $\varepsilon > 0$ there is a $\delta > 0$ such that

If $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ ← distance from (x,y) to (a,b)

Then $|f(x,y) - L| < \varepsilon$

Notice: in order for the limit to exist, we need

$f \to L$ as $(x,y) \to (a,b)$ along <u>any path</u>

Limit along a path

Let $(x(t), y(t))$ be a parametric path so that

$(x(0), y(0)) = (a,b)$

Consider $\displaystyle\lim_{t\to 0} f(x(t), y(t))$

If $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$ then this also $= L$.

Eg. $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ $(a,b) = (0,0)$

Path 1: $x(t) = t$, $y(t) = 0$ x-axis

$f(x(t), y(t)) = \dfrac{t^2}{t^2} = 1$ $\lim_{t\to 0} f(x(t), y(t)) = 1$

Path 2: $x(t) = 0$, $y(t) = t$ y-axis

$f(x(t), y(t)) = \dfrac{-t^2}{t^2} = -1$ $\lim_{t\to 0} f(x(t), y(t)) = -1$

$\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

Eg: $f(x,y) = \dfrac{xy}{x^2+y^2}$

Path 1: $x(t) = t$, $y(t) = 0$

$f(t,0) = \dfrac{t0}{t^2+0^2} = 0$ $\lim_{t\to 0} f(t,0) = 0$

Path 2: $x(t) = 0$ $y(t) = t$

$f(0,t) = \dfrac{0t}{0^2+t^2} = 0$ $\lim_{t\to 0} f(0,t) = 0$

Path 3: $y=x$ $\quad x(t)=t,\ y(t)=t$

$$f(t,t)=\frac{t\cdot t}{t^2+t^2}=\frac{t^2}{2t^2}=\frac{1}{2} \qquad \lim_{t\to 0} f(t,t)=\frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist.}$$

Path 4: $y=mx$ $\quad x(t)=t \quad y(t)=mt$

$$f(t,mt)=\frac{t\cdot mt}{t^2+(mt)^2}=\frac{mt^2}{(1+m^2)t^2}=\frac{m}{1+m^2}$$

Eg: $f(x,y)=\dfrac{xy^2}{x^2+y^4}$

Path $y=mx$ $\quad x(t)=t,\ y(t)=mt$

$$f(t,mt)=\frac{t(mt)^2}{t^2+(mt)^4}=\frac{m^2t^3}{t^2+m^4t^4}$$

$$\lim_{t\to 0}\left(\frac{m^2t^3}{t^2+m^4t^4}\right)=\lim_{t\to 0} t\left(\frac{m^2}{1+m^4t^2}\right)=0$$

Path 2: $y^2=x$ $\quad x(t)=t^2 \quad y(t)=t$

$$f(t^2,t) = \frac{t^2 t^2}{(t^2)^2+(t)^4} = \frac{t^4}{t^4+t^4} = \frac{1}{2}$$

So $\lim\limits_{(x,y)\to(a,b)} \frac{xy^2}{x^2+y^4}$ does not exist

[Checking cleverly chosen paths can show that a limit does not exist, but you can't prove that it does exist.

Positive results:
Limit properties hold:

$$\lim f+g = \lim f + \lim g$$
$$\lim f\cdot g = \lim f \cdot \lim g$$

$$\lim cf = c \lim f$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad \text{if } \lim g \neq 0$$

Squeeze: if $f(x,y) \leq g(x,y) \leq h(x,y)$

and $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \lim\limits_{(x,y)\to(a,b)} h(x,y) = L$

then $\lim\limits_{(x,y)\to(a,b)} g(x,y)$ exists and equals L

Def $f(x,y)$ is continuous at (a,b)
if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equals $f(a,b)$

Ex: A polynomial is continuous

monomial $x, y, x^2, y^2, xy, x^2y, x^5y^{10}, x^my^n$

Polynomial sum of monomials

eg $2x^2+y^2+5x^3y^4+24x$

To take limit of continuous func., just plug in

Rational function $\dfrac{f(x,y)}{g(x,y)}$ where f and g are polynomials

A rational function is continuous everywhere the denominator does not vanish

$$\lim_{(x,y)\to(1,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{1^2-0^2}{1^2+0^2} = 1$$

$$f(x,y) = \frac{3x^2y}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \text{ exists and } = 0$$

$$|f(x,y)| = \left|\frac{3x^2y}{x^2+y^2}\right| = 3|y| \left|\frac{x^2}{x^2+y^2}\right|$$

Notice $\left|\frac{x^2}{x^2+y^2}\right| \leq 1$

$$0 \leq |f(x,y)| \leq 3|y|$$

$$-3y \leq f(x,y) \leq 3y$$

$\downarrow$ 0 $\qquad$ $\downarrow$ 0 $\qquad$ $\downarrow$ 0 as $(x,y) \to 0$

$f(x,y) \to 0$ by squeeze theorem.

Partial Derivative: just as we can take limits in different directions, we can take derivatives in different directions.

Take derivative in x-direction, while holding y constant

$f(x,y)$

$\frac{\partial f}{\partial x} = f_x$ $\qquad$ Partial derivative with respect to x

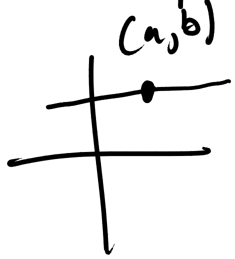
Take deriv. in y-directn, holding x constant

$\frac{\partial f}{\partial y} = f_y$ Partial derivative with respect to y.

There's no reason for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to be equal and they are almost never equal.

- $\frac{\partial f}{\partial x} = f_x$ hold y constant, and vary x.

To compute $f_x(a,b)$ Fix $y=b$ constant.



Then we get $g(x) = f(x,b)$

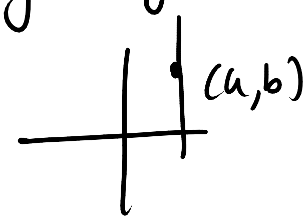
Then take $g'(x)$ deriv w.r.t. x

Plug in a $g'(a)$

$$f_x(a,b) = g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

- $\frac{\partial f}{\partial y} = f_y$ $f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$



Symbolically, just treat one variable as const

$f(x,y) = x^2 - y^2$, find $\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$ at $(x,y) = (1,1)$

$$\frac{\partial f}{\partial x} = 2x - 0$$

b/c y^2 is constant w.r.t. x. $(\text{constant})' = 0$

$$\frac{\partial f}{\partial y} = 0 - 2y$$

$$\frac{\partial f}{\partial x}(1,1) = 2 \qquad \frac{\partial f}{\partial y}(1,1) = -2$$

$$f(x,y) = xy \qquad \frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$$

$$f(x,y) = e^{-y} \cos \pi x$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{-y} \cos \pi x\right) = e^{-y} \frac{\partial}{\partial x}(\cos \pi x)$$

$$= e^{-y}(-\pi \sin \pi x)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{-y} \cos \pi x\right) = \frac{\partial}{\partial y}\left(e^{-y}\right) \cos \pi x$$

$$= -e^{-y} \cos \pi x$$

$w = \ln(u+2v)$ find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$

$$\frac{\partial w}{\partial u} = \frac{\partial}{\partial u} \ln(u+2v) = \frac{1}{u+2v} \underbrace{\frac{\partial}{\partial u}(u+2v)}_{1} = \frac{1}{u+2v}$$

$$\frac{\partial w}{\partial v} = \frac{\partial}{\partial v} \ln(u+2v) = \frac{1}{u+2v} \frac{\partial}{\partial v}(u+2v) = \frac{2}{u+2v}$$

Implicit differentiation works.

$x^2+y^2+z^2=1$ $\quad \frac{\partial z}{\partial x}$ holding y constant

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2=1\right) \Rightarrow 2x+0+2z\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2z} = \frac{-x}{z}$$

Higher derivatives $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Usually,

It doesn't matter what order you take partial derivatives in. $f_{xy} = f_{yx}$

(as long as both continuous)