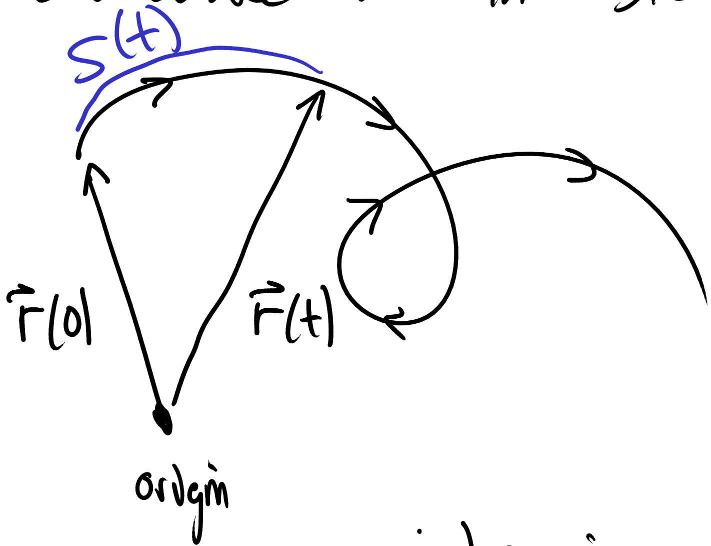
Curvature and intrinsic properties of curves.



Not natural:
- choose an origin
- choose a parametrization

But convenient.

intrinsic = doesn't depend on parametrization or choice of origin.

Arclength doesn't depend on parametrization

$$S = \int_{t_0}^{t_1} |\vec{r}\,'(t)|\,dt = \int_{t_0}^{t} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

do substitution for t $\qquad t = f(u)$

u = new parameter

$\vec{r} = \vec{r}(u)$ $\qquad dt = \frac{dt}{du}\,du$

$$\frac{d}{du}\vec{r} = \frac{d}{dt}\vec{r} \cdot \frac{dt}{du} \qquad \begin{aligned} f(u_0) &= t_0 \\ f(u_1) &= t_1 \end{aligned}$$

$$s = \int_{u_0}^{u_1} \left|\frac{d\vec{r}}{du}\right| du = \int_{u_0}^{u_1} \left|\frac{d\vec{r}}{dt} \cdot \frac{dt}{du}\right| \left(\frac{dt}{du}\right)^{-1} dt$$

$$= \int \left|\frac{d\vec{r}}{dt}\right| dt$$

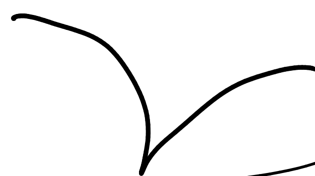
Think of arclength as a function along the curve

$$s(t) = \int_0^t |\vec{r}\,'(t)|\,dt$$

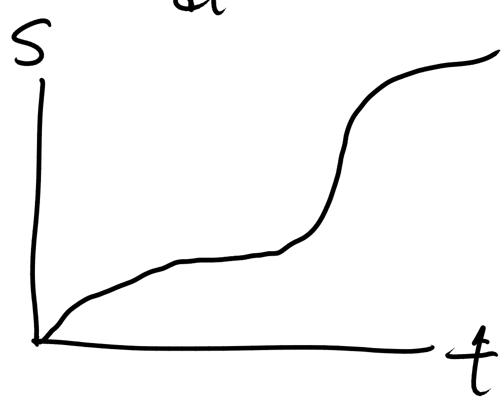
$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \text{speed.} = \text{rate of change of arclength}$$

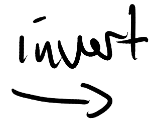
Assume $\vec{r}\,'(t)$ is never zero:

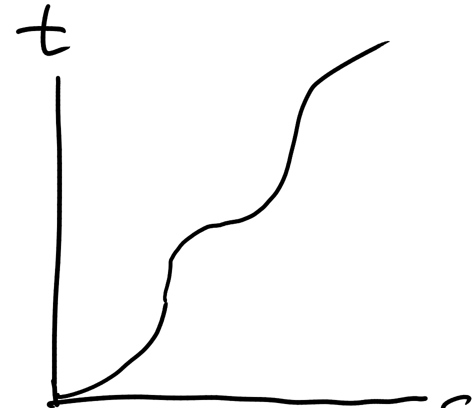
Bad



Then $\frac{ds}{dt}$ is always positive $\Rightarrow$ s is increasing as a function of t.







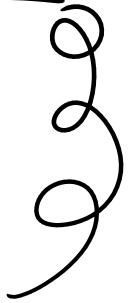
Thus $s(t)$ has an inverse $t(s)$
(Abstractly it exists)

Then, we can use arclength as the parameter

$$\vec{r}(s) = \vec{r}(t(s))$$

This is called <u>parametrization with respect to arclength</u>.

<u>Ex</u> $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ Helix



$$s(t) = \int_0^t |\vec{r}\,'(t)|\,dt$$

$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 1\vec{k}$$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

$$s(t) = \int_0^t \sqrt{2}\,dt = \left[\sqrt{2}t\right]_{t=0}^{t=t} = \sqrt{2}\,t$$

$$t(s) = \frac{s}{\sqrt{2}}$$

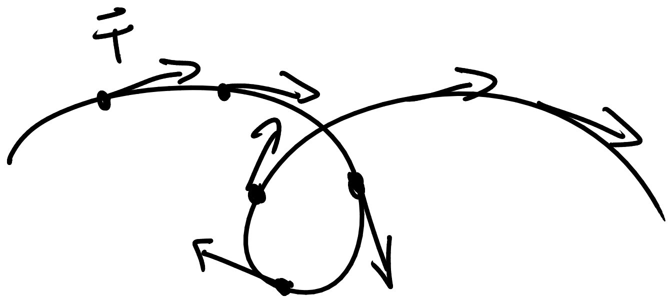
Arclength parametrization: $\vec{r}(s) = \cos\frac{s}{\sqrt{2}}\,\vec{i} + \sin\frac{s}{\sqrt{2}}\,\vec{j} + \frac{s}{\sqrt{2}}\vec{k}$

Parametrization with respect to arclength always has speed $= 1$

$$\left|\frac{d\vec{r}}{ds}\right| = \left|\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}\right| = \left|\frac{d\vec{r}}{dt}\right| \Big/ \left|\frac{ds}{dt}\right| = \frac{|\vec{r}\,'(t)|}{|\vec{r}\,'(t)|} = 1$$

velocity in this parametrization is $\vec{T}$ unit tangent vector:

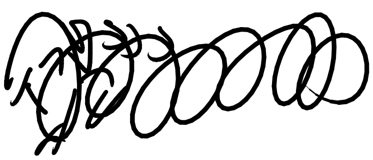
$$\vec{v}(t) = \vec{r}\,'(t) = |\vec{r}\,'(t)|\,\vec{T}(t)$$



$\vec{T}$ always is parallel to the tangent line = line that best approximates the curve.

2nd derivatives can tell us about curvature of the curve. (related to acceleration, but not same.)

straight  $\vec{T}$ is constant.

curvy  $\vec{T}$ changes rapidly.

<u>Curvature</u> kappa $\kappa = \left| \frac{d\vec{T}}{ds} \right|$ ← arclength param.

<u>Note</u>: $\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} = \frac{d\vec{T}}{dt} \Big/ \frac{ds}{dt}$

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}}{dt} \right| \Big/ \left| \frac{ds}{dt} \right| = \left| \frac{d\vec{T}}{dt} \right| \Big/ |\vec{r}\,'(t)|$$

$$\kappa = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$ ← can use any parametrization

$$\left[\vec{T}\,'(t) = \frac{d}{dt}\vec{T} = \frac{d}{dt}\left[\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \right] \right]$$

Find the curvature of $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$$\vec{r}\,'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$$

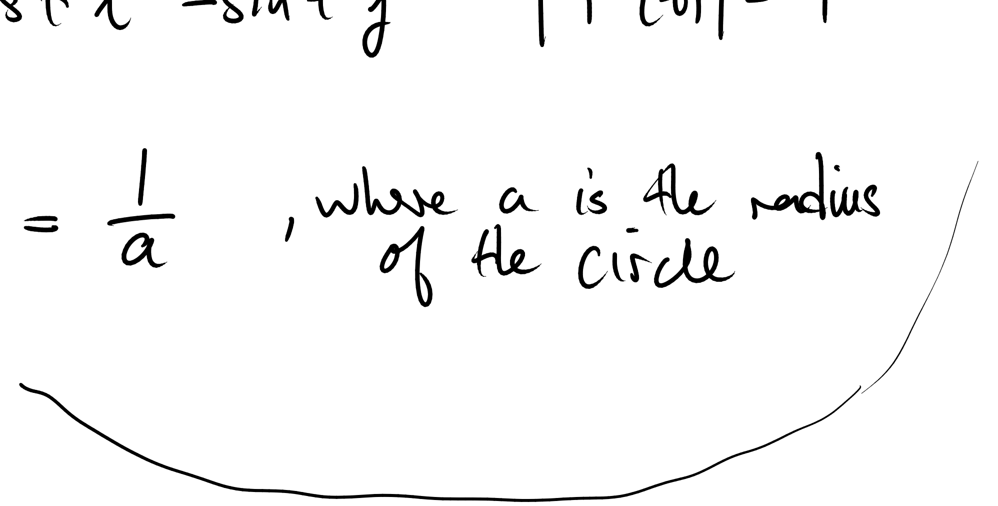
$$|\vec{r}\,'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{-a\sin t\,\vec{i} + a\cos t\,\vec{j}}{a} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\vec{T}\,'(t) = -\cos t\,\vec{i} - \sin t\,\vec{j} \qquad |\vec{T}\,'(t)| = 1$$

$$\kappa = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{a}$$, where a is the radius of the circle

high κ



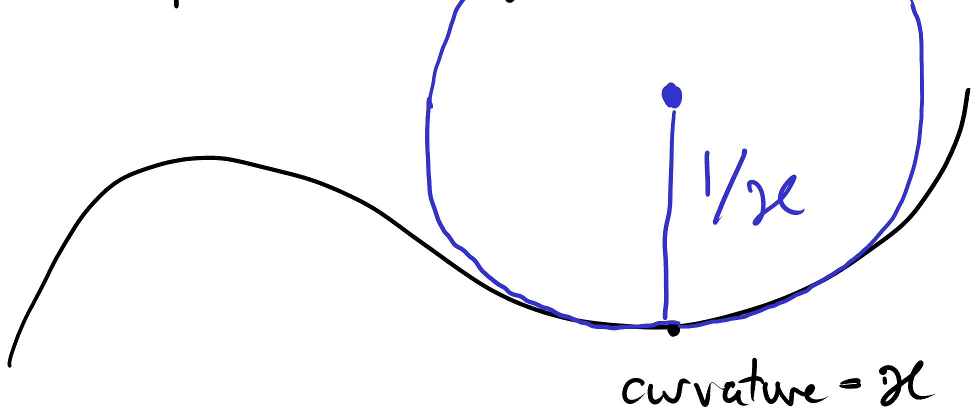
low κ

Starting from $\vec{r}(t)$ in 3Dimensions

$$\boxed{\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}}$$

units $\frac{1}{\text{Length}}$

Interpretation of κ in 2 dimensions



- circle is tangent and has the same curvature
- The circle that approximates the curve best.
- called <u>Osculating</u> <u>circle</u>

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| \qquad \vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

unit vector in direction of $\vec{T}'(t)$

Called <u>normal vector</u>

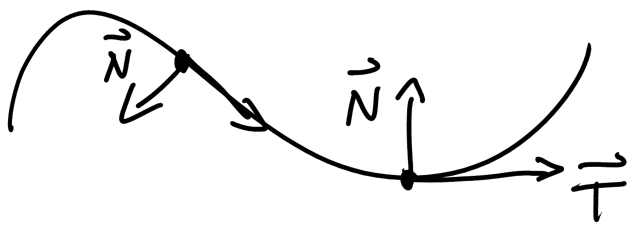
Fact $\vec{T}\cdot\vec{N} = 0$ $\quad$ $\vec{T}$ and $\vec{N}$ are perpendicular

<u>Proof</u> Start with $|\vec{T}| = 1 \iff \vec{T}\cdot\vec{T} = 1$

$$0 = \frac{d}{dt}\vec{T}\cdot\vec{T} = \vec{T}'\cdot\vec{T} + \vec{T}\cdot\vec{T}' = 2\vec{T}\cdot\vec{T}'$$

$\Rightarrow \vec{T}\cdot\vec{T}' = 0$ $\quad$ so $\vec{T}$ and $\vec{T}'$ are <u>perp</u>.

$\vec{T}$ and $\vec{N}$ determine a plane calle the osculating plane



In 3D, get one more vector:

$$\vec{B} = \vec{T} \times \vec{N} \quad \boxed{\text{Binormal}}$$

unit $\quad |\vec{B}| = |\vec{T}||\vec{N}| \sin\left(\frac{\pi}{2}\right) = 1 \cdot 1 \cdot 1 = 1$

$$\vec{B} \cdot \vec{T} = 0, \quad \vec{B} \cdot \vec{N} = 0$$

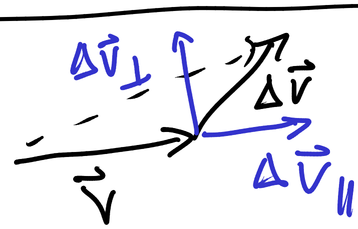
$\vec{T}, \vec{N}, \vec{B}$ are all mutually perpendicular and are unit vectors.

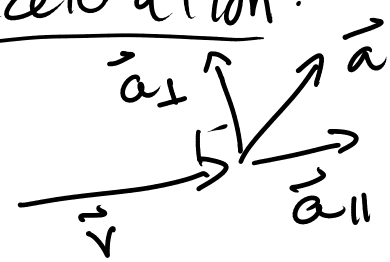
(orthonormal basis)

$\vec{T}, \vec{N}$, and $\vec{B}$ are just like $\vec{i}, \vec{j}, \vec{k}$

This system of vectors is called the TNB frame or the Frenet-Serret frame.

<u>Tangential and normal acceleration:</u>





$$\vec{a} = \vec{a}_\parallel + \vec{a}_\perp$$

$\vec{a}_\parallel$ is parallel to $\vec{v}$

$\vec{a}_\perp$ is perp to $\vec{v}$

$$\vec{a}_{\parallel} = \mathrm{proj}_{\vec{v}}\, \vec{a} \qquad \vec{a}_{\perp} = \vec{a} - \vec{a}_{\parallel}$$

$$\vec{v} = \vec{r}\,' = |\vec{r}\,'|\, \vec{T}$$

$$\vec{a} = \frac{d}{dt}\vec{v} = \left(\frac{d}{dt}|\vec{r}\,'(t)|\right)\vec{T} + |\vec{r}\,'(t)|\, \vec{T}\,'$$

$$= \left(\frac{d}{dt}|\vec{r}\,'|\right)\vec{T} + |\vec{r}\,'|\,|\vec{T}\,'|\,\vec{N}$$

$$= \left(\frac{d}{dt}|\vec{r}\,'|\right)\vec{T} + |\vec{r}\,'|^2 \kappa\, \vec{N}$$

$$= \left(\frac{d}{dt}|\vec{v}|\right)\vec{T} + |\vec{v}|^2 \kappa\, \vec{N}$$

$$\vec{a}_{\parallel} = \left(\frac{d}{dt}|\vec{v}|\right)\vec{T} \qquad \vec{a}_{\perp} = |\vec{v}|^2 \kappa\, \vec{N}$$