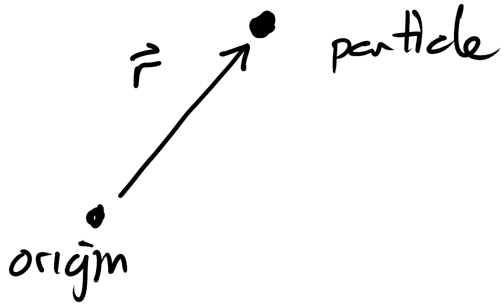
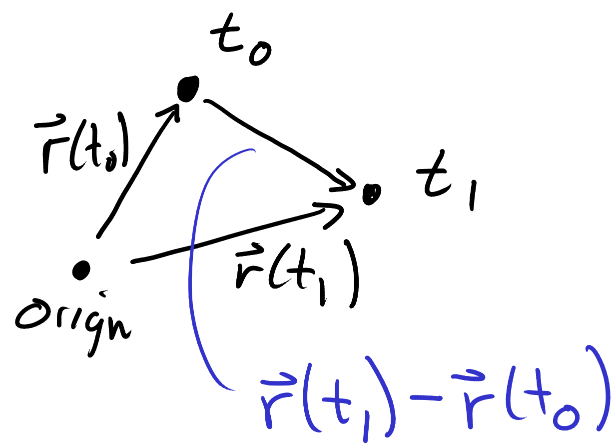
Velocity and Acceleration (Kinematics)

$t_0 < t_1$

Position:





displacement vector between t_0 and t_1

· Displacement doesn't depend on what origin you pick.

Average velocity = $\dfrac{\text{displacement}}{\text{time}}$

$$\vec{V}_{average, t_0, t_1} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}$$

units Length/time

instantaneous velocity = limit of average velocities as time interval goes to zero.

$$\vec{V} = \lim_{t_1 \to t_0} \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0} = \lim_{h \to 0} \frac{\vec{r}(t_0 + h) - \vec{r}(t_0)}{h}$$

$(h = t_1 - t_0)$

$$\vec{V}(t) = \vec{r}\,'(t) = \frac{d\vec{r}}{dt}$$

Geometrically, $\vec{v}(t)$ is a tangent vector to the path the particle moves along

derivative of velocity is acceleration

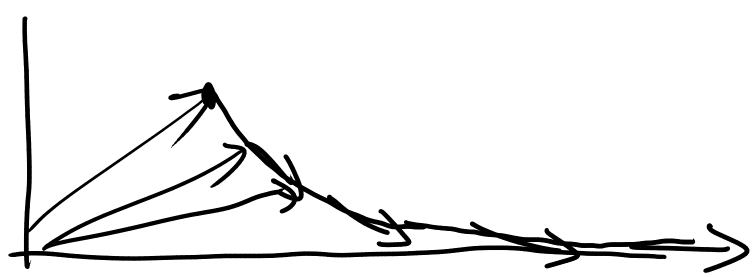
$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \quad \text{(instantaneous acceleration)}$$

<u>Ex</u> $\vec{r}(t) = A\left(e^{\alpha t}\vec{i} + e^{-\alpha t}\vec{j}\right)$

Find $\vec{v}(t) = A\left(\alpha e^{\alpha t}\vec{i} - \alpha e^{-\alpha t}\vec{j}\right)$

$\alpha > 0$

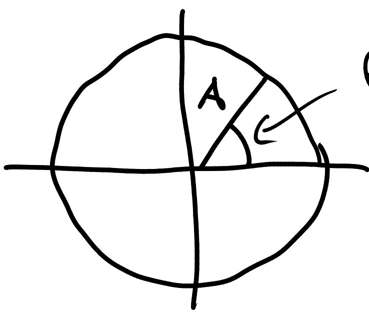
$A > 0$



Uniform circular motion

$$\vec{r}(t) = A(\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j})$$

ω = omega = angular frequency



$\theta = \omega t$

Let T = period = time for a complete rotation

complete rotation: $\boxed{2\pi = \omega T}$

f = ordinary frequency = # rotations per unit time

$f = \frac{1}{T}$ $\quad 2\pi = \frac{\omega}{f}$ $\quad 2\pi f = \omega$ $\quad \boxed{f = \frac{\omega}{2\pi}}$

$$\vec{r}(t) = A(\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j})$$

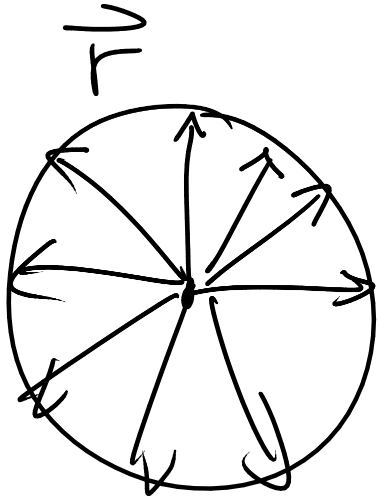
$$\vec{v}(t) = A(-\omega\sin\omega t\,\vec{i} + \omega\cos\omega t\,\vec{j})$$

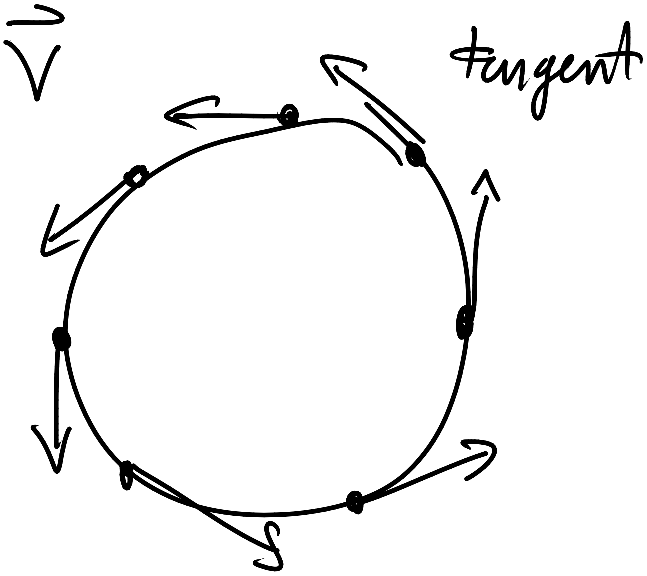
$$= A\omega(-\sin\omega t\,\vec{i} + \cos\omega t\,\vec{j})$$

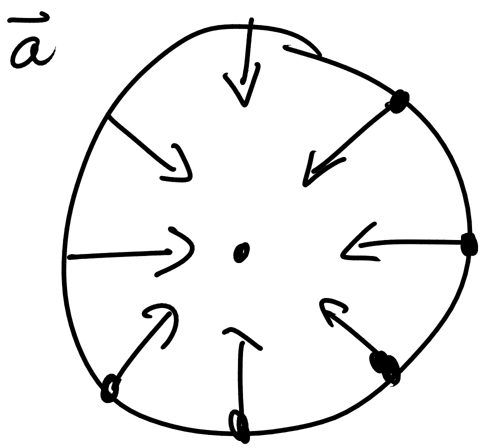
$$\vec{a}(t) = A\omega(-\omega\cos\omega t\,\vec{i} - \omega\sin\omega t\,\vec{j})$$

$$= -A\omega^2(\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j})$$

$\vec{a} = -\omega^2\vec{r}$ for all t







$\vec{a}$ points toward center

$\vec{a}$ = centripetal acceleration

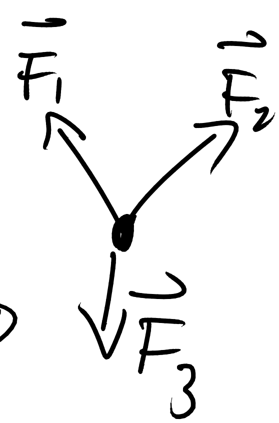
(coming from centripetal force)

Newton's 2nd law ⇐ can write it as a vector equation

$\vec{F} = m\vec{a}$ $\quad$ m = mass $\quad$ $\vec{a}$ = acceleration

$\vec{F}$ = total force acting

Forces add like vectors



$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$ ⟶

If you know the forces, you can determine the motion.

$\vec{F} = m\vec{a} \iff \vec{a} = \frac{\vec{F}}{m}$ Know Force $\Rightarrow$ know $\vec{a}$

Know $\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow$ compute $\vec{v}$ by integration

$$\vec{v}(t_1) - \vec{v}(t_0) = \int_{t_0}^{t_1} \frac{d\vec{v}}{dt}\, dt = \int_{t_0}^{t_1} \vec{a}(t)\, dt$$

$$\vec{v}(t_1) = \vec{v}(t_0) + \int_{t_0}^{t_1} \vec{a}(t)\, dt$$

if you know initial velocity $\vec{v}(t_0)$ and $\vec{a}(t)$, then you know $\vec{v}(t_1)$ for any other value of t_1

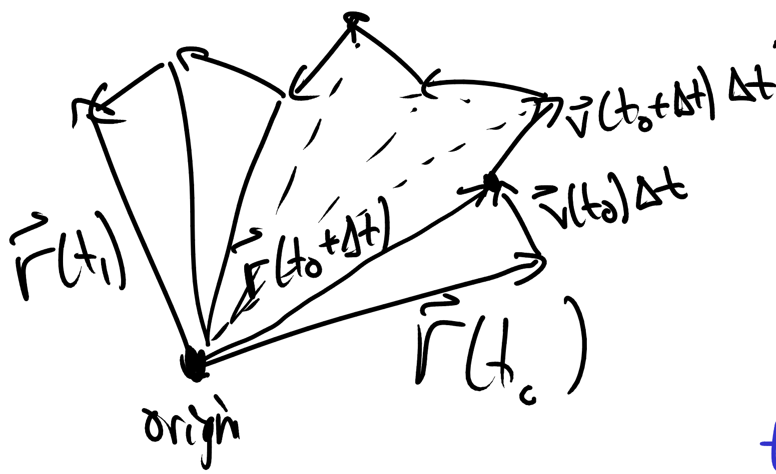
if $\vec{v}_0$ = initial velocity

$$\left.\vec{v}(t) = \vec{v}_0 + \int_{t_0}^{t} \vec{a}(t')\, dt' \quad \right\}\text{ another notation.}$$

Same process yields $\vec{r}$ from $\vec{v}$, once we know initial position $\vec{r}_0 = \vec{r}(t_0)$

$$\vec{r}(t_1) - \vec{r}(t_0) = \int_{t_0}^{t_1} \frac{d\vec{r}}{dt}\, dt = \int_{t_0}^{t_1} \vec{v}(t)\, dt$$

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{v}(t')\, dt'$$



Δt = small interval of time

t' = dummy variable

<u>Example</u>: No acceleration: $\vec{a}(t) = 0$ initial position $\vec{r}_0$ initial velocity $\vec{v}_0$

(Assume $t_0 = 0$)

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t')\,dt' = \vec{v}_0 + \int 0\,dt = \vec{v}_0$$

$$\vec{r}(t) = \vec{r}_0 + \int_0^{t} \vec{v}(t')\,dt' = \vec{r}_0 + \int_0^t \vec{v}_0\,dt'$$

$$= \vec{r}_0 + \vec{v}_0 \int_0^t dt' = \vec{r}_0 + \vec{v}_0 \left[t'\right]_{t'=0}^{t'=t}$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$ $\leftarrow$ a line $\vec{r}_0$ = point on line, $\vec{v}_0$ = direction vector.

- <u>Constant acceleration</u> $\vec{a}(t) = \vec{a}$

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(s)\,ds = \vec{v}_0 + \int_0^t \vec{a}\,ds = \vec{v}_0 + \vec{a}t$$

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(s)\,ds = \vec{r}_0 + \int_0^t (\vec{v}_0 + \vec{a}s)\,ds$$

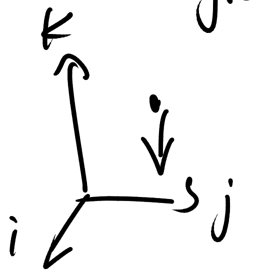
$$= \vec{r}_0 + \int_0^t \vec{v}_0 \, ds + \int_0^t \vec{a} s \, ds$$

$$= \vec{r}_0 + \vec{v}_0 \int_0^t ds + \vec{a} \int_0^t s \, ds$$

$$= \vec{r}_0 + \vec{v}_0 t + \left(\frac{1}{2} t^2\right) \vec{a}$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2$$

Constant gravitational acceleration $\vec{a} = -g\vec{k}$



$g = 9.8\,m/s^2$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}(-g\vec{k})t^2$$

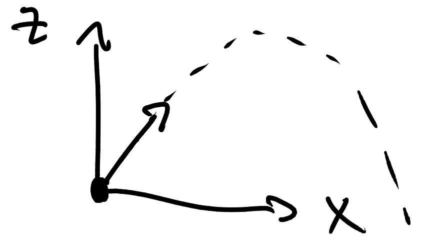
$\vec{r}_0 = \langle r_{0x}, r_{0y}, r_{0z} \rangle \quad \vec{V}_0 = \langle v_{0x}, v_{0y}, v_{0z} \rangle$

$x = r_{0x} + v_{0x} t$

$y = r_{0y} + v_{0y} t$

$z = r_{0z} + v_{0z} t - \frac{1}{2} g t^2$

Assume $\vec{r}_0 = 0$, $v_{0y} = 0$



parabolic trajectory.

Non constant acceleration: electron e^-

charge $-e$ mass m

Electron is subjected to an electrical field

$$\vec{E} = \vec{E}_0 \sin \omega t \qquad \vec{a} = \frac{-e\vec{E}}{m}$$

$\vec{E}_0$: constant vector.

$$\vec{a}_0 = \frac{-e\vec{E}_0}{m}$$

$$\vec{a}(t) = \frac{-e\vec{E}_0}{m} \sin \omega t = \vec{a}_0 \sin \omega t$$

Assume $\vec{v}_0 = 0 \quad \vec{r}_0 = 0 \quad t_0 = 0$

$$\vec{v}(t) = \int_0^t \vec{a}(s)\,ds = \int_0^t \vec{a}_0 \sin \omega s\,ds$$

$$= \vec{a}_0 \left[-\frac{1}{\omega} \cos \omega s \right]_0^t = -\frac{\vec{a}_0}{\omega} \left[\cos \omega t - 1 \right]$$

$$\vec{r}(t) = \int_0^t \vec{v}(s)\,ds = \int_0^t -\frac{\vec{a}_0}{\omega} \left[\cos \omega s - 1 \right] ds$$

$$= -\frac{\vec{a}_0}{\omega} \left[\frac{1}{\omega} \sin \omega s - s \right]_0^t = -\frac{\vec{a}_0}{\omega} \left[\frac{1}{\omega} \sin \omega t - t \right]$$

$$\vec{r}(t) = \underbrace{\frac{\vec{a}_0}{\omega} t}_{\text{drift.}} - \underbrace{\frac{\vec{a}_0}{\omega^2} \sin \omega t}_{\text{oscillation}}$$