Calculus of Vector functions

Different parameterizations of a space curve.

$\vec{r}(t) = \langle t, t^2, t^3 \rangle \qquad 1 \le t \le 2$

path from $\vec{r}(1) = \langle 1, 1, 1 \rangle$
to $\vec{r}(2) = \langle 2, 4, 8 \rangle$

t = parameter

$\vec{r}(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \le u \le \ln 2$

path from $\vec{r}(u=0) = \langle 1, 1, 1 \rangle$
to $\vec{r}(u=\ln 2) = \langle 2, 4, 8 \rangle$

u = parameter

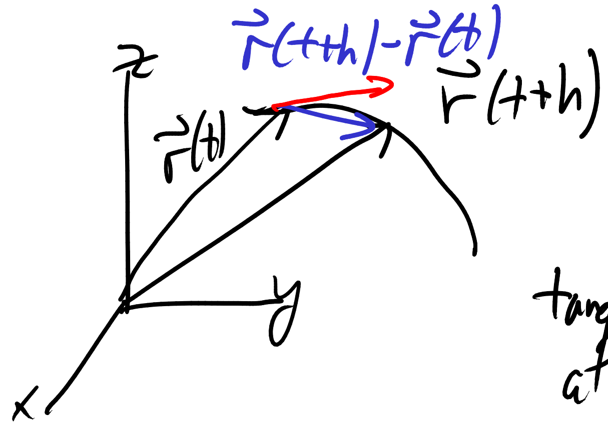
These two curves are geometrically the same, but they have different parameterization.

Related by change of variable $\quad t = e^u$
$u = \ln t$

Derivatives $\quad$ vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

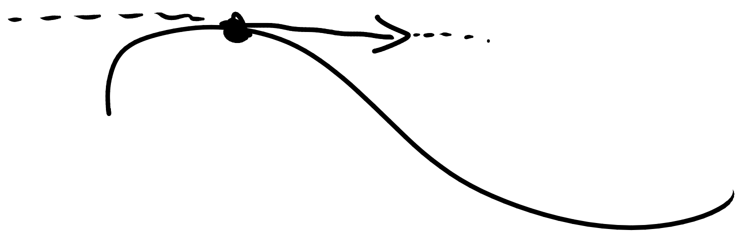
Picture



$\rightarrow = \vec{r}\,'(t)$

$\vec{r}\,'(t)$ is a vector tangent to the curve at at the point $\vec{r}(t)$

In 1-var: $f'(t)$ = slope of tangent line
In 3-d: $\vec{r}\,'(t)$ = tangent vector



The limit definition is equivalent to taking derivatives of components:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\lim_{\varepsilon \to 0} \frac{\vec{r}(t+\varepsilon) - \vec{r}(t)}{\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \left\langle \frac{f(t+\varepsilon) - f(t)}{\varepsilon}, \frac{g(t+\varepsilon) + g(t)}{\varepsilon}, \frac{h(t+\varepsilon) - h(t)}{} \right\rangle$$

$$= \left\langle \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon) - f(t)}{\varepsilon}, \lim_{\varepsilon \to 0} \frac{g(t+\varepsilon) - g(t)}{\varepsilon}, \lim_{\varepsilon \to 0} \frac{h(t+\varepsilon) - h(t)}{\varepsilon} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

$\vec{r}\,'(t)$ = tangent vector = velocity vector.

direction: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ unit tangent vector

This tells us what direction the curve is moving at an instant in time.

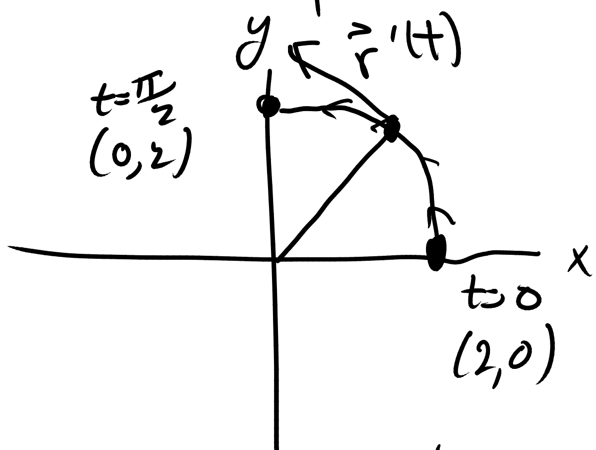
works in 2D as well.

Eg $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = \sqrt{4(\cos^2 t + \sin^2 t)} = \sqrt{4} = 2$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{r}\,'(t)}{2} = \langle -\sin t, \cos t \rangle$$



$t=\frac{\pi}{4}$ $\vec{r}(t) = \langle \sqrt{2}, \sqrt{2} \rangle$

$$\vec{r}\,'(t) = \langle -\sqrt{2}, \sqrt{2} \rangle$$

$$\vec{T}(t) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

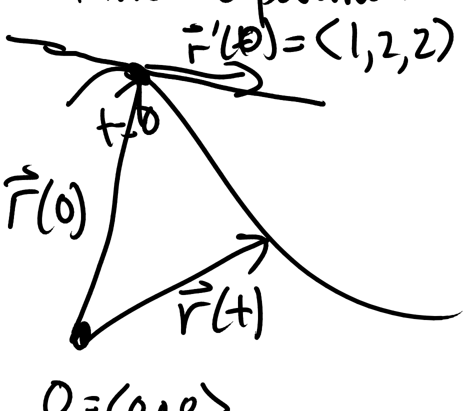
slope of tangent line $\left(\text{at } t = \frac{\pi}{4}\right) = -1 = \dfrac{y'(t)}{x'(t)}$

$$= \frac{\sqrt{2}}{(-\sqrt{2})} = \frac{1/\sqrt{2}}{(-1/\sqrt{2})}$$

Ex(3D)

$$\vec{r}(t) = \langle te^{-t}, 2\arctan t, 2e^{t} \rangle$$

Find a parametric equation for tangent line at $t=0$



$$\vec{r}\,'(t) = \left\langle e^{-t} + t(-e^{-t}), \frac{2}{1+t^2}, 2e^{t} \right\rangle$$

plug in $t=0$

$$\vec{r}\,'(0) = \langle 1, 2, 2 \rangle$$

This is the direction vector of tangent line.

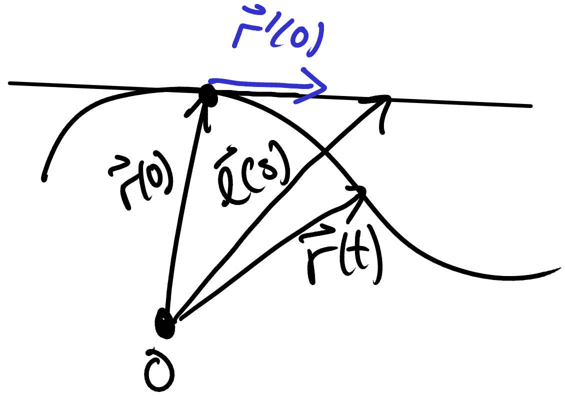
just need a point on the tangent line.
just take $\vec{r}(0)$ as the point.

$$\vec{r}(0) = \langle 0, 0, 2 \rangle$$

Let $\vec{l}(s)$ be a vector function describing the tangent line (s = parameter)

$$\vec{l}(s) = \vec{r}(0) + s\,\vec{r}\,'(0)$$

$$\vec{l}(s) = \langle 0,0,2\rangle + s\langle 1,2,2\rangle = \langle s, 2s, 2+2s\rangle$$



Higher derivatives $\quad \vec{r}\,''(t) = \left(\vec{r}\,'(t)\right)'$

$\vec{r}\,'''(t) = \left(\vec{r}\,''\right)'$

$\vec{r}(t)$ = position (based at origin)

$\vec{r}\,'(t)$ = tangent vector/velocity (based at $\vec{r}(t)$)

$\vec{r}\,''(t)$ = acceleration vector (based at $\vec{r}(t)$)

$\vec{r}\,'''(t) = ?$

Derivative rules

Linearity: $\frac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \vec{u}'(t)+\vec{v}'(t)$

$$\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}'(t)$$

Product rules: $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

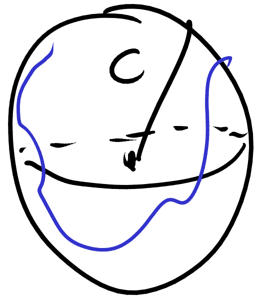
$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \underbrace{\vec{u}(t)\times\vec{v}'(t)}_{\text{order matters}}$$

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

Chain rule:

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = \vec{u}'(f(t))\,f'(t)$$

Proposition: Suppose the path $\vec{r}(t)$ lies on a sphere of radius C centered at the origin. then $\vec{r}'(t)$ is perpendicular to $\vec{r}(t)$.



Know $|\vec{r}(t)| = C$

$$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = C^2$$

Take derivation $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \frac{d}{dt}\left[C^2\right] = 0$

$$\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$$

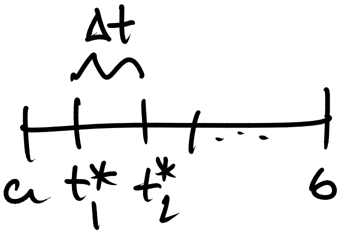
$$= 2\,\vec{r}\,'(t)\cdot\vec{r}(t)$$

$$2\,\vec{r}\,'(t)\cdot\vec{r}(t) = 0 \Rightarrow \vec{r}\,'(t)\cdot\vec{r}(t) = 0$$

$\Leftrightarrow$ $\vec{r}\,'(t)$ is perpendicular to $\vec{r}(t)$

Integration of vector valued functions.

$$\int_a^b \vec{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{r}(t_i^*)\,\Delta t$$



Riemann sum

Equivalent to integrating one component at a time

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

FTC: If $\vec{R}\,'(t) = \vec{r}(t)$ then $\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$

Indefinite integral $\int \vec{r}(t)dt$ means (any) antiderivatives

<u>Ex</u> $\int (\cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k})\,dt$

$$= (\sin t + C_1)\vec{i} + (-\cos t + C_2)\vec{j} + (\tfrac{1}{2}t^2 + C_3)\vec{k}$$
$$= \sin t\,\vec{i} - \cos t\,\vec{j} + \tfrac{1}{2}t^2\vec{k} + C_1\vec{i} + C_2\vec{j} + C_3\vec{k}$$
$$= (\sin t\,\vec{i} - \cos t\,\vec{j} + \tfrac{1}{2}t^2\vec{k}) + \vec{C}$$

$$\int_0^{2\pi} (\cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k})\,dt$$
$$= \left[\sin t\,\vec{i} - \cos t\,\vec{j} + \tfrac{1}{2}t^2\vec{k}\right]_0^{2\pi} = \tfrac{1}{2}(2\pi)^2\vec{k}$$
$$= \langle 0, 0, \tfrac{1}{2}(2\pi)^2 \rangle$$

<u>Arclength</u>: in 2D $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

In 3D $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

Arclength $= \int (\text{speed})\, d(\text{time})$

speed (of parametric curve $\vec{r}(t)$) $= |\vec{r}\,'(t)|$

speed $= |\text{velocity}|$

$$\text{speed} = |\vec{r}\,'(t)| = |\langle x'(t), y'(t), z'(t)\rangle|$$
$$= \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Arclength : $L = \int_a^b |\vec{r}\,'(t)|\, dt$

Ex Helix $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$

$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 1\,\vec{k}$$
$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

Arclength from $t=0$ to $t=2\pi$

$$L = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\cdot 2\pi$$