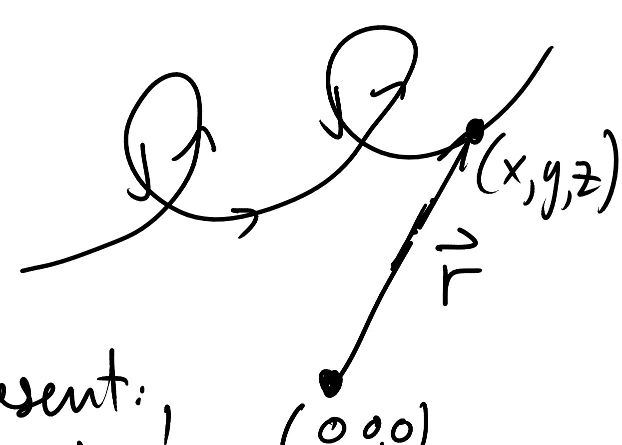
Vector functions and Space Curves

Vector-valued function $\vec{r}(t)$ input number t
output vector $\vec{r}$

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

t = parameter : Can think of $\vec{r}(t)$ as describing a parameterized curve in 3-dimensional space

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$



If 3rd component not present:
2-dimensional parametric curve!

limits of $\vec{r}$:

$$\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} \langle f(t), g(t), h(t) \rangle$$

$$= \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

(one component at a time)

Find: $\lim_{t \to 0} \left(e^{-3t} \vec{i} + \frac{t^2}{\sin^2 t} \vec{j} + \cos 2t \, \vec{k} \right)$

$$\lim_{t \to 0} e^{-3t} = e^{-3 \cdot 0} = e^0 = 1$$

$$\lim_{t \to 0} \frac{t^2}{\sin^2 t} = \lim_{t \to 0} \frac{2t}{2 \sin t \cos t} = \lim_{t \to 0} \frac{2t}{\sin 2t}$$

$$= \lim_{t \to 0} \frac{2}{2 \cos 2t} = \frac{2}{2 \cos 0} = 1$$

$$\lim_{t \to 0} \cos 2t = \cos 0 = 1$$

$$\lim_{t \to 0} \vec{r}(t) = 1\vec{i} + 1\vec{j} + 1\vec{k} = \vec{i} + \vec{j} + \vec{k} = \langle 1, 1, 1 \rangle$$

Continuity: $\vec{r}(t)$ is continuous at a if $\vec{r}(a)$ exists and $\vec{r}(a) = \lim_{t \to a} \vec{r}(t)$

same as requiring that all components are continuous at a

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

We've seen one example of a vector function

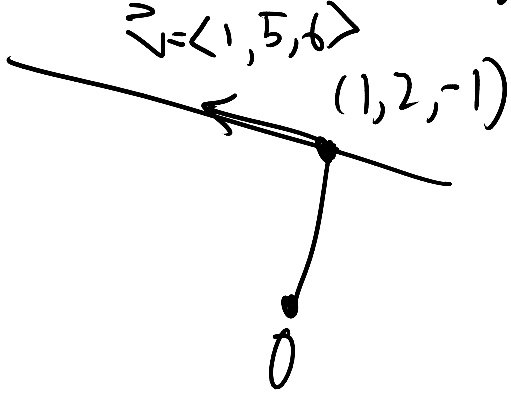
$\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$

Parametrized line in 3d.

Goes through $(1,2,-1)$ (set $t=0$)

also through $(2,7,5)$ (set $t=1$)

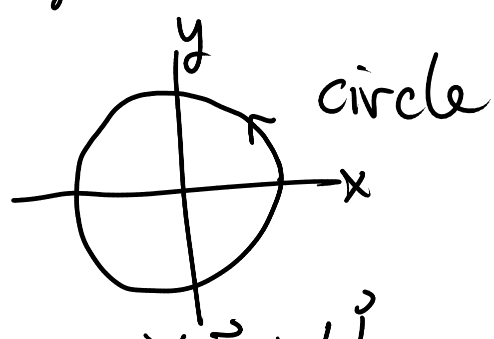
Direction vector $\vec{v} = \langle 1,5,6 \rangle$



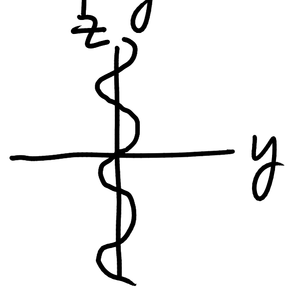
$\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\vec{k} = \langle \cos t, \sin t, t \rangle$

Forget about z for a second: get $\cos t\,\vec{i} + \sin t\,\vec{j}$

Project to xy-plane



Forget about x keep y and z : $\sin t\,\vec{j} + t\vec{k}$

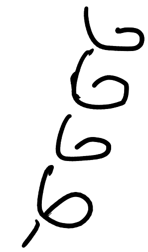


$y = \sin t$

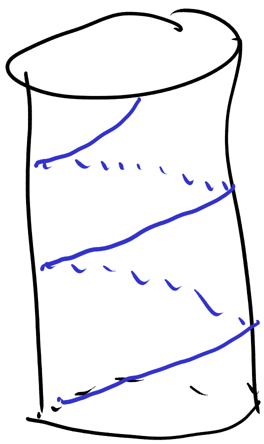
$z = t$

$y = \sin z$

Put it together: get a helix



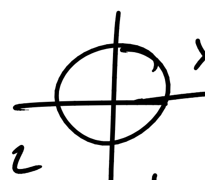
Observe, $\vec{r}(t)$ lies on the surface $x^2+y^2=1$



- Show that $\vec{r}(t) = \langle \sin t, \cos t, \sin^2 t \rangle$

  lies on the surfaces $z = x^2$ and $x^2+y^2=1$

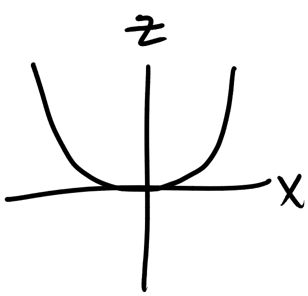
xy-projection $x = \sin t$ $y = \cos t$



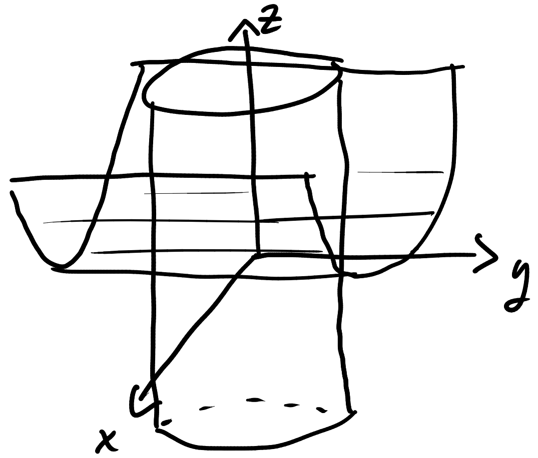
$x^2+y^2=1$

Or just compute $x^2+y^2 = (\sin t)^2 + (\cos t)^2 = 1$

For $z = x^2$ $x = \sin t$ $z = \sin^2 t$ $z = \sin^2 t = (\sin t)^2 = x^2$



The parametric curve is the intersection of the two surfaces



Take $y+z=2$ plane and $x^2+y^2=1$ cylinder

Describe the intersection as a parametric curve.

Intuitively, we expect it to be an ellipse.

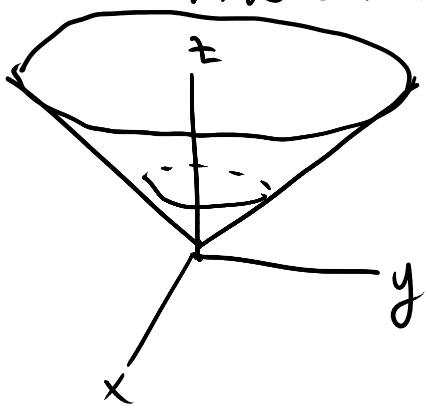
$y = 2 - z \qquad x^2 + (2-z)^2 = 1 \; \ldots$

Parametrize circle $x^2+y^2=1$ $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

$z = 2 - y = 2 - \sin t$

Answer : $\vec{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$

Parameterize intersection of $z = \sqrt{x^2+y^2}$ cone and $z = 1 + y$ plane



$z^2 = x^2 + y^2$

$(1+y)^2 = x^2 + y^2$

$1 + 2y + y^2 = x^2 + y^2$

$1+2y = x^2 \qquad y = \frac{x^2-1}{2}$

Hierarchy $x \longrightarrow y$, $(x,y) \to z$

once you know x, know y and z as well

$x = t$ (use x as parameter)

$y = \frac{x^2-1}{2} = \frac{t^2-1}{2}$

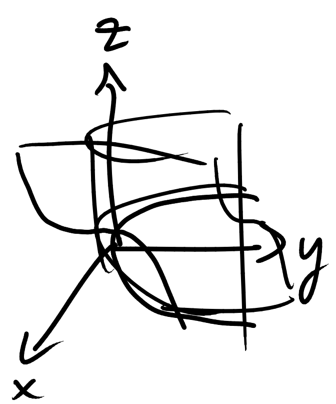
$z = \sqrt{x^2+y^2} = \sqrt{t^2 + \left(\frac{t^2-1}{2}\right)^2}$ $\qquad z = 1+y = \frac{t^2+1}{2}$

$\vec{r}(t) = \left\langle t, \frac{t^2-1}{2}, \sqrt{t^2+\left(\frac{t^2-1}{2}\right)^2} \right\rangle$

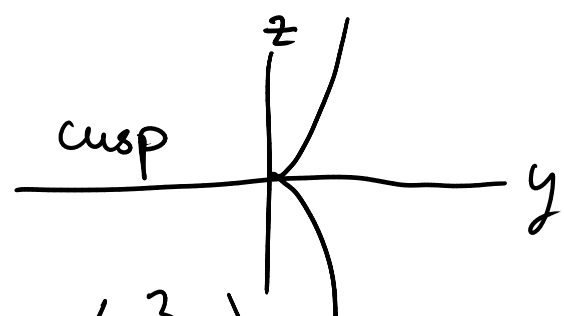
$= \left\langle t, \frac{t^2-1}{2}, \frac{t^2+1}{2} \right\rangle$

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$$y^3 = z^2 \quad \left\langle \begin{array}{l} x = t \\ y = t^2 \\ z = t^3 \end{array} \right\rangle \quad \begin{array}{l} y = x^2 \\ z = x^3 \end{array}$$



$y = z^{2/3}$



$(y-x^2) + 2(x^3-z) = 0$