Cross products (3D only)

$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{c} = \vec{a} \times \vec{b}$ is a vector called cross product

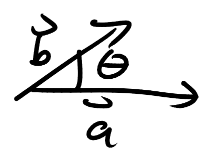
$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2 ,\ a_3 b_1 - a_1 b_3 ,\ a_1 b_2 - a_2 b_1 \rangle$$

more complicated than dot product.

No cross product for 2D vectors.

Geometric: $\vec{a} \times \vec{b}$ is a vector < magnitude, direction

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

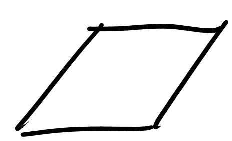


Contrast $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

Direction is perpendicular to both $\vec{a}$ and $\vec{b}$

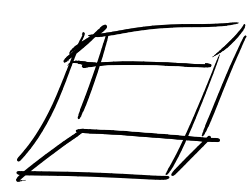
orientation is determined by right-hand rule

Uses: Areas of parallelogram



Volumes of parallelopiped

(opposite faces parallelogram)



Finding equations of planes in 3D

Physics: Torque (rotational mechanics)

Electricity and magnetism.

<u>Algebra</u>: $\vec{i}\ \vec{j}\ \vec{k}$ notation

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ 3x3 determinant

2x2 determinant $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC$

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

\ has + sign / has − sign

3x3 determinant

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = A_1 \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$- A_2 \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} + A_3 \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= A_1 \begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - A_2 \begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + A_3 \begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

2x2 so use formula

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \vec{i}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$$

Another way to compute

$$\begin{array}{ccc:cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array}$$

$\diagdown$ have + signs

$\diagup$ have − signs

$\vec{a} = \langle 1, 3, 4 \rangle \qquad \vec{b} = \langle 2, 7, -5 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$

$$= \vec{i}\big((3)(-5) - (4)(7)\big) - \vec{j}\big(1(-5) - 4 \cdot 2\big)$$

$$+ \vec{k}\big(1 \cdot 7 - 3 \cdot 2\big) = -43\vec{i} + 13\vec{j} + \vec{k}$$

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 3 & 4 & 1 & 3 \\ 2 & 7 & -5 & 2 & 7 \end{array}$$

$$\vec{i}(3)(-5) + \vec{j}(4)(2) + \vec{k}(1)(7) - \vec{k}(3)(2) - \vec{i}(4)(7) - \vec{j}(1)(-5)$$

$$= -43\vec{i} + 13\vec{j} + \vec{k}$$

<u>Algebraic properties</u>: $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

(In particular $\vec{a}\times\vec{a} = -\vec{a}\times\vec{a} \Rightarrow 2(\vec{a}\times\vec{a}) = 0$

so $\vec{a}\times\vec{a} = 0$.

Distributive $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

Scalar $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b})$

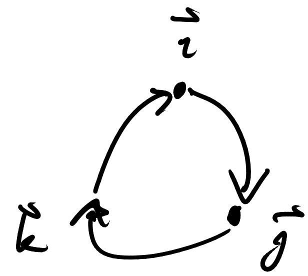
$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$ Not associative.

$$\vec{i}\times(\vec{i}\times\vec{j}) = \vec{i}\times\vec{k} = -\vec{j}$$

$$(\vec{i}\times\vec{i})\times\vec{j} = \vec{0}\times\vec{j} = 0$$

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \qquad -\vec{j}$$

Basis vector $\vec{i}\ \vec{j}\ \vec{k}$



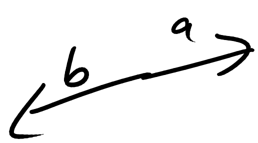
$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$$

$$\vec{i}\times\vec{j}=\vec{k}\qquad \vec{j}\times\vec{k}=\vec{i}\qquad \vec{k}\times\vec{i}=\vec{j}$$

$$\vec{j}\times\vec{i}=-\vec{k}\qquad \vec{k}\times\vec{j}=-\vec{i}\qquad \vec{i}\times\vec{k}=-\vec{j}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\qquad \vec{b}\times\vec{a}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

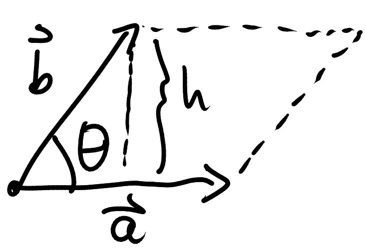
$\vec{a}\times\vec{b}=0$ if and only if $\vec{a}$ and $\vec{b}$ are parallel



<u>Geometry</u> length/magnitude $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$

(p. 810 has a proof)

<u>Observation</u>: $|\vec{a}||\vec{b}|\sin\theta$ = Area of parallelogram spanned by $\vec{a}$ and $\vec{b}$



base $=|\vec{a}|$ $\qquad \sin\theta=\dfrac{h}{|\vec{b}|}$

height $=|\vec{b}|\sin\theta$

Area = base · height $=|\vec{a}||\vec{b}|\sin\theta=|\vec{a}\times\vec{b}|$

Area of parallelogram spanned by $\vec{a} = \langle 1,3,0 \rangle$
$\vec{b} = \langle 2,1,10 \rangle$

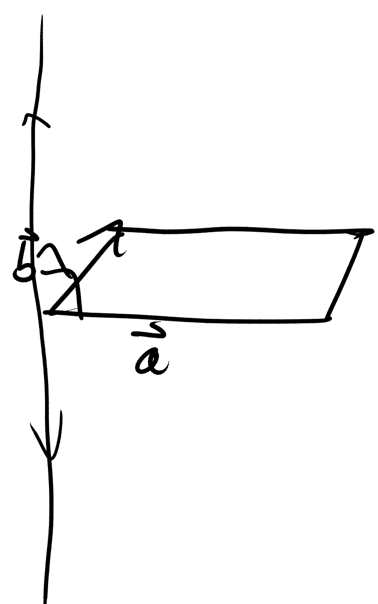
$A = |\vec{a} \times \vec{b}|$ $\quad \vec{a} \times \vec{b} = \langle 30, -10, -5 \rangle$ from determinant

$A = \sqrt{(30)^2 + (-10)^2 + (-5)^2} = \sqrt{1025}$

Direction $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$

Proof $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ (direct computation)

$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$



right hand rule

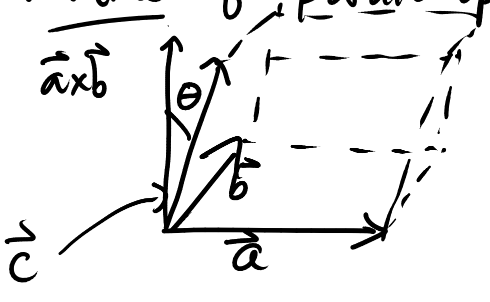
For $\vec{a} \times \vec{b}$

curl fingers from $\vec{a}$ to $\vec{b}$

Thumb = direction of $\vec{a} \times \vec{b}$

(Do Not use left hand)

Volume of parallelopiped spanned by $\vec{a}, \vec{b}, \vec{c}$



V = Area of base $\times$ height

$= |\vec{a} \times \vec{b}| \, |\vec{c}| \cos\theta$

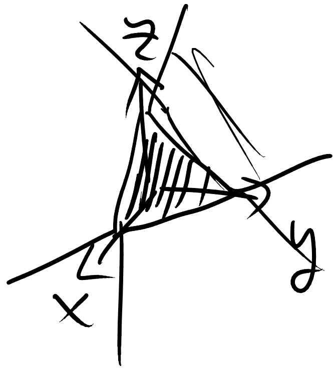
$$V = \left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right| \quad \boxed{\begin{aligned} &= \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right| \\ &= \left| \vec{b} \cdot (\vec{a} \times \vec{c}) \right| \end{aligned}}$$

Dont worry about order just take absolute value

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \leftarrow \text{might be negative.}$$

Planes in 3D $\leftrightarrow$ linear equation $ax + by + cz = d$



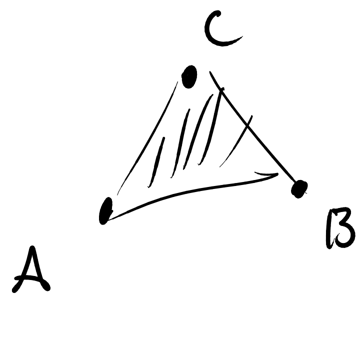
$$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = d$$

$$\langle x, y, z \rangle = \vec{r} \quad \text{position of general point.}$$

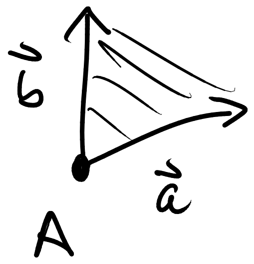
$$\langle a, b, c \rangle = \vec{n} \quad \text{normal vector of the plane.}$$

$\vec{n}$ is perpendicular to the plane.

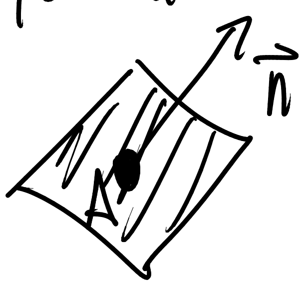
What determines a plane. 3 points

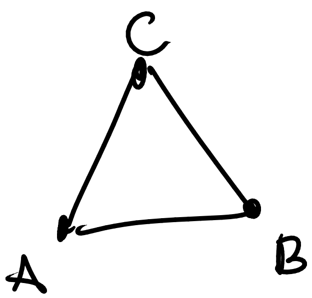


or 1 point and two vectors parallel to the plane

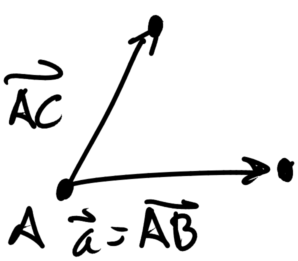


or 1 point and one vector perpendicular to the plane

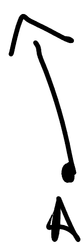




$\rightsquigarrow \vec{b} = \vec{AC}$



$\rightsquigarrow$



$\vec{n} = \vec{a} \times \vec{b} = \vec{AB} \times \vec{AC}$

$A = (3, -1, 2)$ $\quad\quad \vec{a} = \vec{AB} = \langle 5, 3, 2 \rangle$

$B = (8, 2, 4)$ $\quad\quad \vec{b} = \vec{AC} = \langle -4, -1, -5 \rangle$

$C = (-1, -2, -3)$ $\quad\quad \vec{a}$ & $\vec{b}$ parallel to plane.

$\vec{n} = \vec{a} \times \vec{b} = \langle -13, 17, 7 \rangle$

$$-13x + 17y + 7z = d$$

plug in coordinates of A to figure out d.

$$-13(3) + 17(-1) + 7(2) = d$$

Can plug in any point on the plane to find d

$$-39 - 17 + 14 = d$$

$$-39 - 3 = d \qquad d = -42$$

$$\boxed{-13x + 17y + 7z = -42}$$

Other equations differing by an overall factor are also correct eg

$$\frac{13}{42}x - \frac{17}{42}y - \frac{7}{42}z = 1$$ also correct.