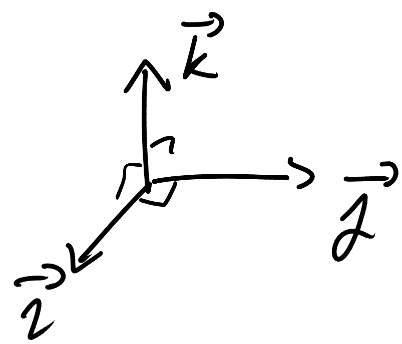
Dot product, equations of lines

recall 3D vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

$\vec{i} = \langle 1, 0, 0 \rangle$
$\vec{j} = \langle 0, 1, 0 \rangle$
$\vec{k} = \langle 0, 0, 1 \rangle$



(Algebraic)
<u>Definition</u> $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

Dot product $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

e.g. $\langle 4, 1, \frac{1}{4} \rangle \cdot \langle 6, -3, -8 \rangle$

$= 4 \cdot 6 + 1(-3) + \frac{1}{4}(-8) = 24 - 3 - 2 = 19$

2D $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$

<u>Example</u> $(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (\,\overset{0\vec{i}}{\wedge}\, 2\vec{j} - \vec{k})$

$\langle 1, 2, -3 \rangle$ $\langle 0, 2, -1 \rangle$

$= 1 \cdot 0 + 2 \cdot 2 + (-3)(-1) = 0 + 4 + 3 = 7$

Algebraic properties (similar to ordinary mult.)

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ $\qquad \vec{0} = \langle 0,0,0 \rangle$

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$\vec{0} \cdot \vec{a} = 0$

$(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

(c scalar (i.e. number))

$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$

$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$ $\qquad i = \langle 1,0,0 \rangle$, $j = \langle 0,1,0 \rangle$

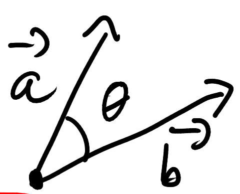
Most useful: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ magnitude / length

$\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2$

relates lengths and dot products

Geometric interpretation of Dot product



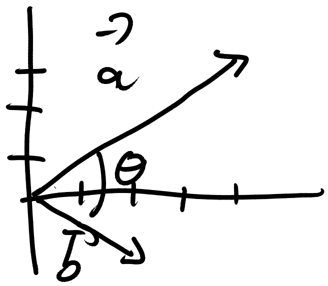
θ = shortest angle between $\vec{a}$ and $\vec{b}$

(in the plane containing $\vec{a}$ and $\vec{b}$)

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta}$$

Using this can compute angles using dot products

$\vec{a} = \langle 4, 3 \rangle \quad \vec{b} = \langle 2, -1 \rangle$ what is the angle between?



$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

$$\vec{a} \cdot \vec{b} = 4 \cdot 2 + 3(-1) = 8 - 3 = 5$$

$$|\vec{a}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

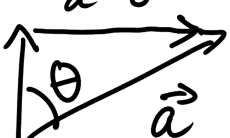
$$|\vec{b}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$5 = 5\sqrt{5}\cos\theta \qquad \cos\theta = \frac{1}{\sqrt{5}}$$

$$\theta \approx 1.107 \text{ radians}$$

<u>Proof</u>: that $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$|\vec{b}| = B$



$|\vec{a}| = A$

compute $|\vec{a} - \vec{b}|^2$ in 2 ways.

use law of cosines

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}|\cos\theta$$

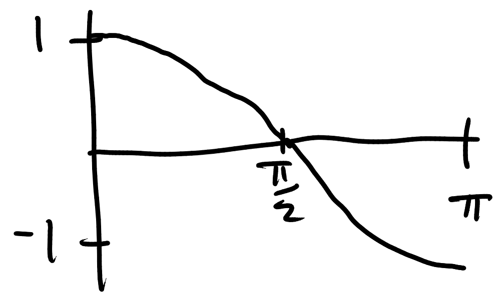
$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$

$$= \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$$

compare and conclude $-2|\vec{a}||\vec{b}|\cos\theta = -2\vec{a}\cdot\vec{b}$

cancel -2 QED.

Look at graph of $\cos\theta$



Acute
$0 \le \theta < \frac{\pi}{2} \Rightarrow \cos\theta > 0$

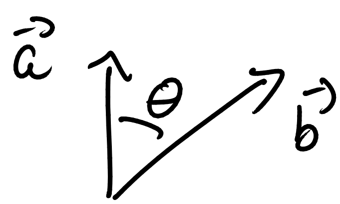
right angle
$\theta = \frac{\pi}{2} \Rightarrow \cos\theta = 0$

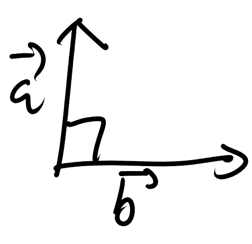
Obtuse $\frac{\pi}{2} < \theta \le \pi \Rightarrow \cos\theta < 0$

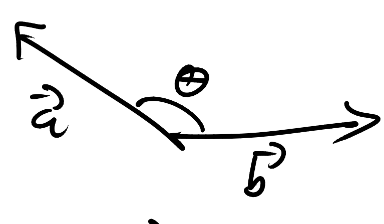
Assume $\vec{a}$ and $\vec{b}$ are not $= \vec{0} = \langle 0,0,0 \rangle$

Since $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ and $|\vec{a}| > 0$ and $|\vec{b}| > 0$

$\Rightarrow$ $\vec{a}\cdot\vec{b}$ has same sign as $\cos\theta$







$\vec{a}\cdot\vec{b} > 0$ $\quad$ $\underbrace{\vec{a}\cdot\vec{b} = 0}$ $\quad$ $\vec{a}\cdot\vec{b} < 0$

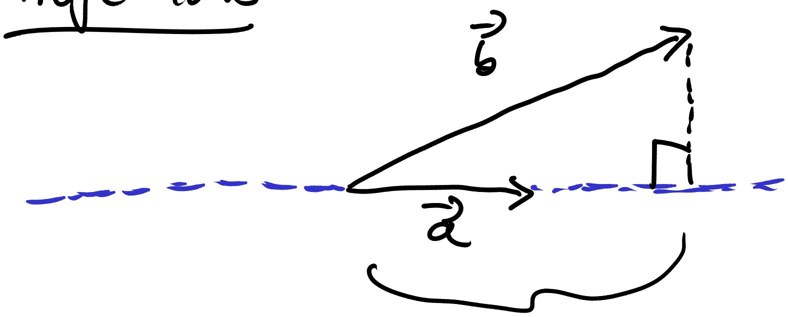
$\hookrightarrow$ this means that $\vec{a}$ and $\vec{b}$ are perpendicular precisely when $\vec{a}\cdot\vec{b} = 0$

$\vec{a} = \langle 2, 1, 12 \rangle \quad \vec{b} = \langle -2, 4, 0 \rangle$

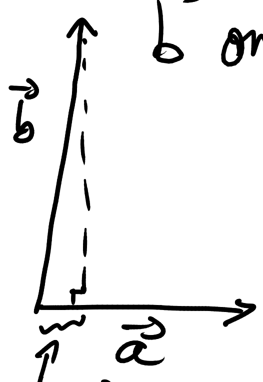
$\vec{a} \cdot \vec{b} = 2 \cdot (-2) + 1(4) + 12 \cdot (0) = -4 + 4 + 0 = 0$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular.

<u>Projections</u>:



this length is the (scalar) projection of $\vec{b}$ onto $\vec{a}$.



$\text{comp}_{\vec{a}} \vec{b}$

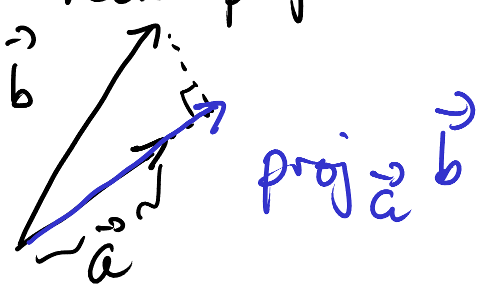
also called component of $\vec{b}$ along $\vec{a}$

$\text{comp}_{\vec{a}} \vec{b}$

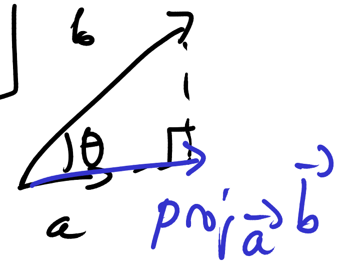
"length of the shadow cast by $\vec{b}$ on the the line parallel to $\vec{a}$"

$\text{comp}_{\vec{a}} \vec{b}$ = how much of $\vec{b}$ points in the direction of $\vec{a}$.

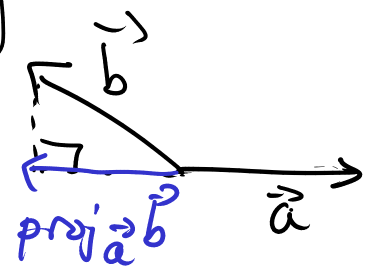
$\text{proj}_{\vec{a}} \vec{b}$ = vector projection



$\boxed{\theta \text{ acute}}$



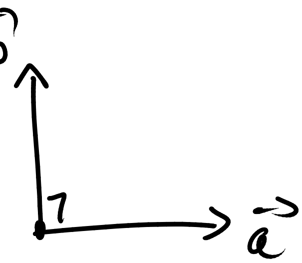
$\boxed{\theta \text{ obtuse}}$



$$\text{comp}_{\vec{a}}\vec{b} > 0 \qquad\qquad \text{comp}_{\vec{a}}\vec{b} < 0$$

$$\text{comp}_{\vec{a}}\vec{b} = |\text{proj}_{\vec{a}}\vec{b}| \qquad\qquad \text{comp}_{\vec{a}}\vec{b} = -|\text{proj}_{\vec{a}}\vec{b}|$$

$\boxed{\theta \text{ right}}$

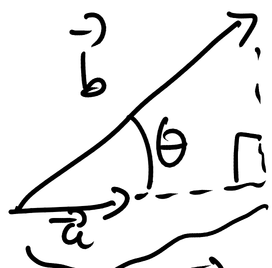


$$\text{comp}_{\vec{a}}\vec{b} = 0$$

$$\text{proj}_{\vec{a}}\vec{b} = 0$$

$\text{comp}_{\vec{a}}\vec{b}$ is a number $(>0, =0, \text{ or } <0)$

$\text{proj}_{\vec{a}}\vec{b}$ is a vector parallel to $\vec{a}$ (but possibly pointing in opposite direction)



$\text{comp}_{\vec{a}}\vec{b}$ is this length $= |\vec{b}|\cos\theta$

$$\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta = \frac{|\vec{a}|}{|\vec{a}|}|\vec{b}|\cos\theta = \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{a}|}$$

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

(unit vector in the direction of $\vec{a}$) $= \dfrac{\vec{a}}{|\vec{a}|}$

$\dfrac{\vec{a}}{|\vec{a}|}$ has magnitude 1 and same direction as $\vec{a}$

$$\operatorname{proj}_{\vec{a}} \vec{b} = \left(\operatorname{comp}_{\vec{a}} \vec{b}\right) \frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|}$$

$$\boxed{\operatorname{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right) \vec{a}}$$

Eg. $\vec{a} = \langle 3, 6, -2 \rangle \qquad \vec{b} = \langle 1, 2, 3 \rangle$

$$\vec{a} \cdot \vec{b} = 3 \cdot 1 + 6 \cdot 2 + (-2) \cdot 3 = 3 + 12 - 6 = 9$$

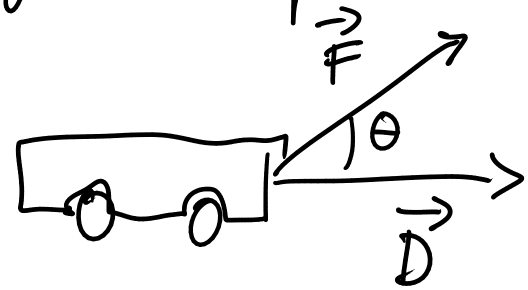
$$|\vec{a}| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

$$\operatorname{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{9}{7}$$

$$\operatorname{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right) \vec{a} = \left(\frac{9}{49}\right) \langle 3, 6, -2 \rangle$$

$$= \left\langle \frac{27}{49}, \frac{54}{49}, \frac{-18}{49} \right\rangle$$

physical example: Work = change in energy

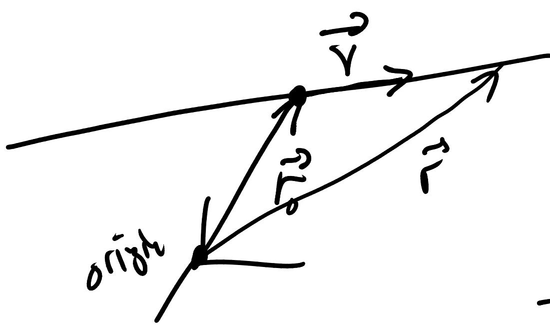


work = (component of force parallel to displacement) × (displacement)

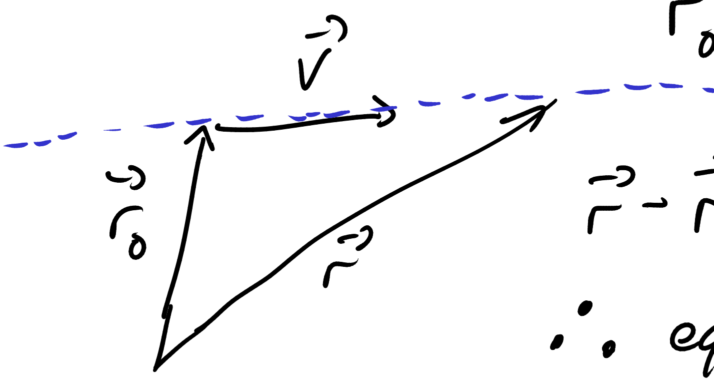
$$W = \left(\text{comp}_{\vec{D}}\, \vec{F}\right) |\vec{D}|$$
$$= \left(\frac{\vec{D}\cdot\vec{F}}{|\vec{D}|}\right) |\vec{D}| = \vec{D}\cdot\vec{F}$$

Equations of lines: in 3D to determine a line:

need • the direction of the line
and • at least one point on the line



$\vec{v}$ = direction vector (a vector parallel to the line)



$\vec{r}_0$ = position vector of some point on the line.

$\vec{r} - \vec{r}_0$ is parallel to $\vec{v}$

$\therefore$ equals $t\vec{v}$ for some # t.

$\therefore \quad \vec{r} - \vec{r}_0 = t\vec{v}, \quad \vec{r} = \vec{r}_0 + t\vec{v}$

Eg. line thru $(1,2,3)$ with direction $\vec{v} = \langle 5,1,3 \rangle$

$$\vec{r} = \langle 1,2,3 \rangle + t\langle 5,1,3 \rangle$$

$$\vec{r} = \langle 1+5t, 2+t, 3+3t \rangle$$

This is a parametric equation for the line.