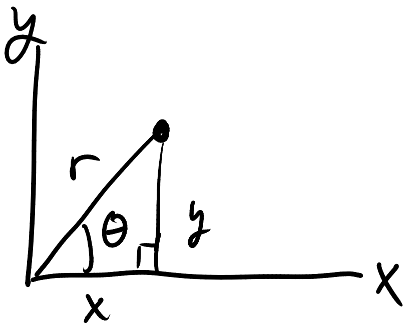
Polar Tangents & Arclengths / Conic sections

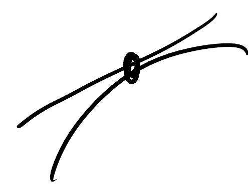
Polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

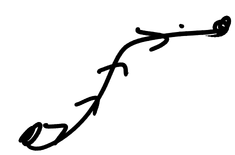


For parametric curve $(x(t), y(t))$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{y'(t)}{x'(t)}$$



$$\text{Arclength} = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



Suppose polar curve is given as $r = f(\theta)$

e.g. $r = 2\cos\theta$, $r = 1 + \sin\theta$

If $r = f(\theta)$:

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases} \leftarrow \text{parametric equations with parameter } \theta.$$

Everything we said about parametric equations applies with θ in place of t

$$\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$$

Slope of tangent line $\dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta + f(\theta)(-\sin\theta)$$

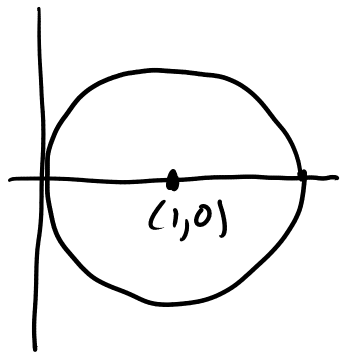
$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Eg. $r = 2\cos\theta = f(\theta)$

$x = r\cos\theta = 2\cos\theta\cos\theta = 2\cos^2\theta$

$y = r\sin\theta = 2\cos\theta\sin\theta$



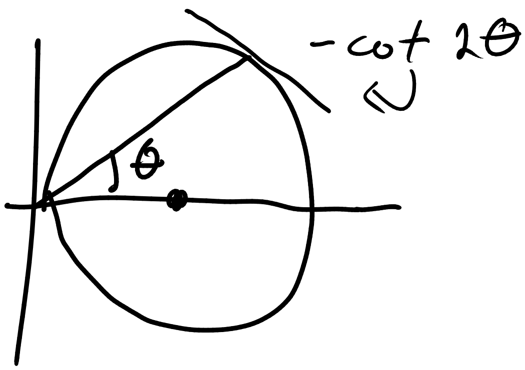
$$\frac{dx}{d\theta} = \frac{d}{d\theta}(2\cos^2\theta) = 4\cos\theta(-\sin\theta)$$
$$= -4\sin\theta\cos\theta = -2\sin 2\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(2\cos\theta\sin\theta) = -2\sin\theta\sin\theta + 2\cos\theta\cos\theta$$

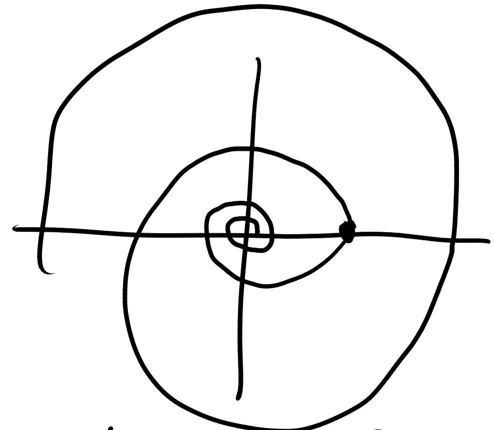
$$\frac{d}{d\theta}(\sin 2\theta) = 2\cos 2\theta$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{2}\cos 2\theta}{-\cancel{2}\sin 2\theta} = -\cot 2\theta = \frac{-1}{\tan 2\theta}$$



Eg. $r = e^{\theta}$



Where is the tangent line horizontal?

$$\frac{dy}{dx} = 0 = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \qquad \text{Need to solve } \frac{dy}{d\theta} = 0$$

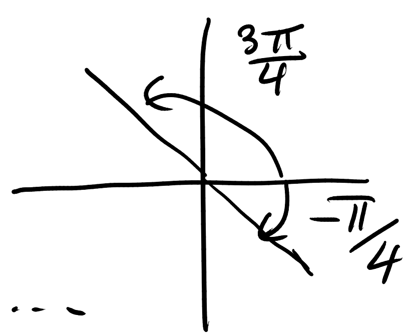
$$y = r\sin\theta = e^{\theta}\sin\theta$$

$$\frac{dy}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta = 0$$

$$e^{\theta}\sin\theta = -e^{\theta}\cos\theta$$

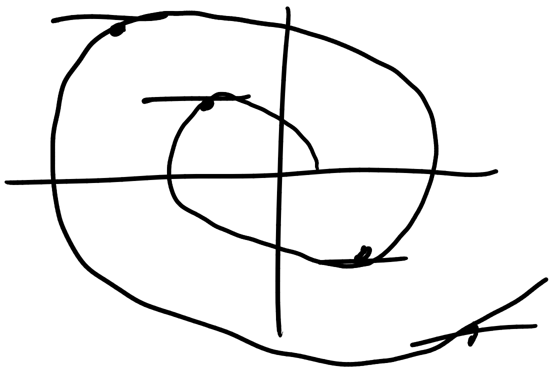
$$\sin\theta = -\cos\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -1$$



All angles $\frac{3\pi}{4}, \frac{3\pi}{4}+\pi, \frac{3\pi}{4}+2\pi, \dots$

Have tangent $= -1$ $\quad \left\{ \frac{3\pi}{4} + n\pi \;\middle|\; n \text{ is an integer.} \right\}$



Arclength:
$$r = f(\theta)$$
$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta$$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$
$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2$$
$$= f'(\theta)^2\cos^2\theta - 2f(\theta)f'(\theta)\cos\theta\sin\theta$$
$$+ f(\theta)^2\sin^2\theta$$

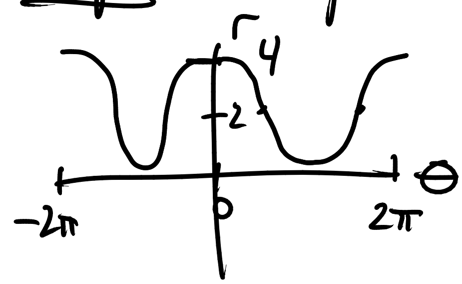
$$\left(\frac{dy}{d\theta}\right)^2 = f'(\theta)^2\sin^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta$$
$$+ f(\theta)^2\cos^2\theta$$

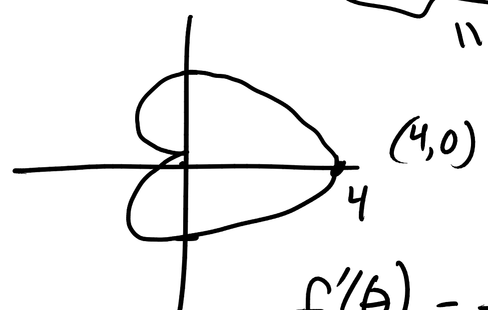
$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2 = f'(\theta)^2(\cos^2\theta+\sin^2\theta)$$
$$+f(\theta)^2(\cos^2\theta+\sin^2\theta)$$
$$=(f'(\theta))^2+f(\theta)^2$$

$$L=\int_\alpha^\beta \sqrt{[f'(\theta)]^2+[f(\theta)]^2}\,d\theta$$

$r=f(\theta)$ $$L=\int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta$$

Example: length of cardioid $r=2\underbrace{(1+\cos\theta)}_{\text{"}f(\theta)}$





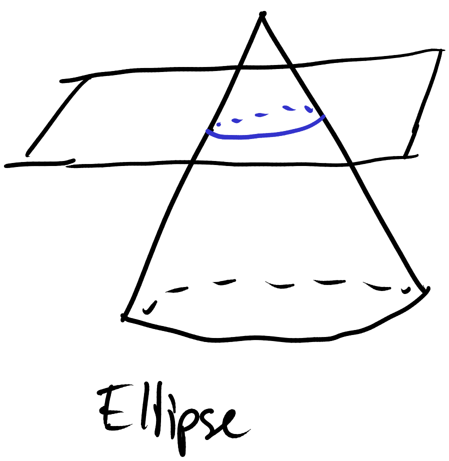
$f'(\theta)=-2\sin\theta$

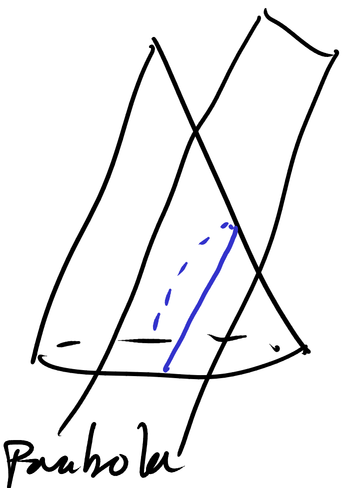
$$L=\int_\alpha^\beta \sqrt{f'(\theta)^2+f(\theta)^2}\,d\theta$$
$$=\int_0^{2\pi}\sqrt{(-2\sin\theta)^2+(2(1+\cos\theta))^2}\,d\theta$$
$$=\int_0^{2\pi}\sqrt{4\sin^2\theta+4(1+2\cos\theta+\cos^2\theta)}\,d\theta$$

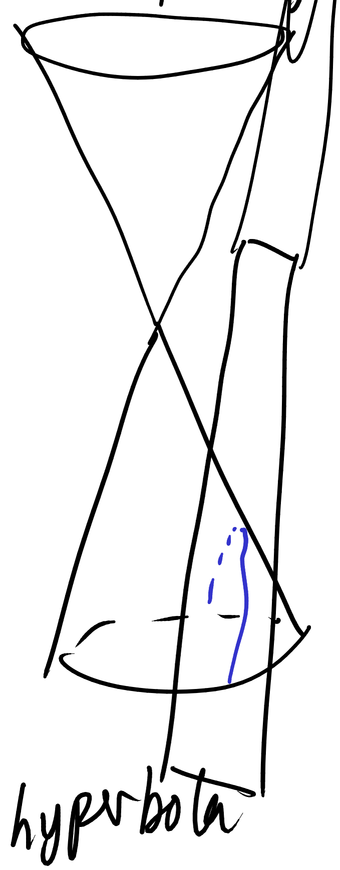
$$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta$$

<u>Conic sections</u>: (preparation for quadric surfaces)

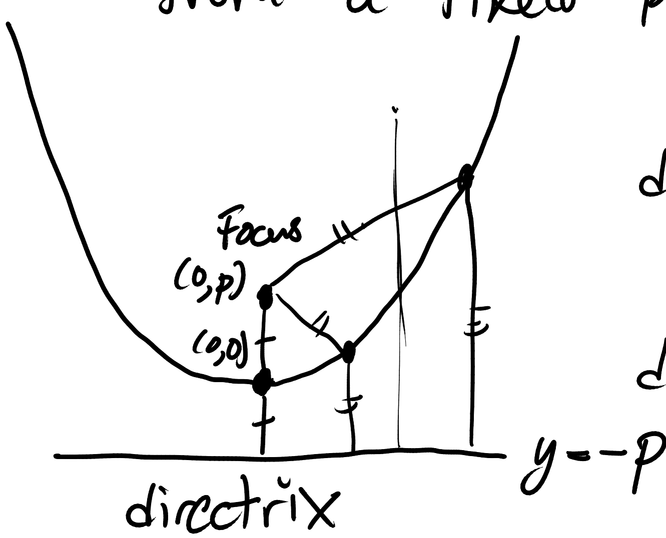
Ellipse (contains circles)
Parabola
Hyperbola







<u>2D</u>: Parabola: set of points that are equidistant from a fixed point and a fixed line.



distance from (x,y) to $\{y=-p\}$

$$= |y+p|$$

distance from (x,y) to $(0,p)$

$$= \sqrt{(x-0)^2 + (y-p)^2}$$

$$= \sqrt{x^2 + (y-p)^2}$$

on the parabola

$$|y+p| = \sqrt{x^2+(y-p)^2}$$

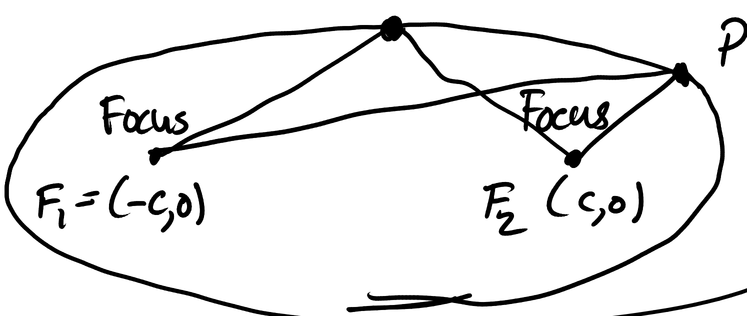
$$(y+p)^2 = x^2+(y-p)^2$$

$$\cancel{y^2} + 2yp + \cancel{p^2} = x^2 + \cancel{y^2} - 2yp + \cancel{p^2}$$

$$4py = x^2$$

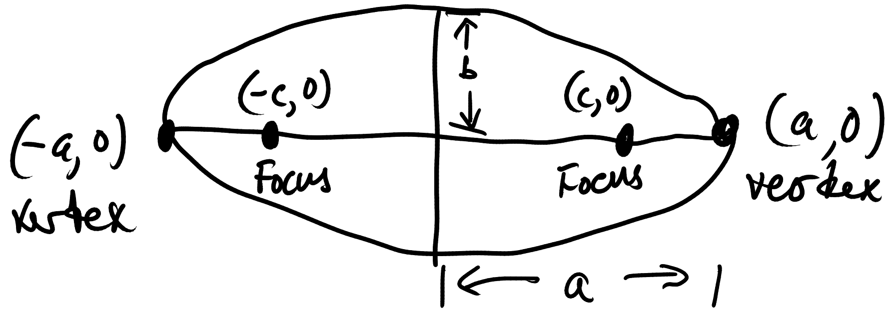
$$y = \left(\frac{1}{4p}\right)x^2 \quad \leftarrow \text{Cartesian equation of a parabola.}$$

<u>Ellipse</u> set of points where the sum of the distances to two fixed points is constant.



$$|F_1P| + |F_2P| = 2a$$

$$\Rightarrow \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

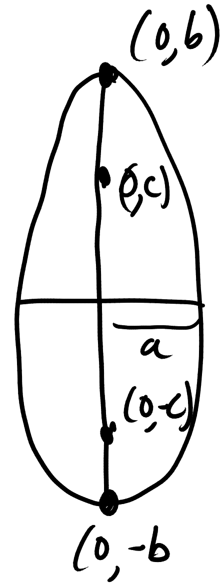


Fact: $b^2 = a^2 - c^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(in this case $b \le a$)

If $a < b$ $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

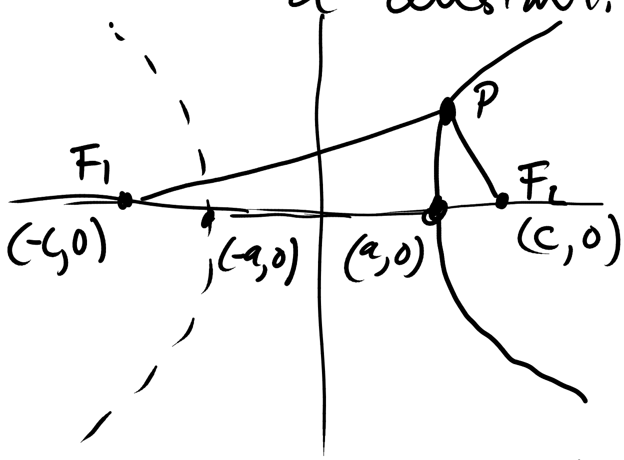


In this case $a^2 = b^2 - c^2$

Parameterize the ellipse

$$\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases}$$

Hyperbola Set of points where difference of distances to two fixed points is a constant.



← this one branch of the hyperbola

$$|PF_1| - |PF_2| = 2a$$

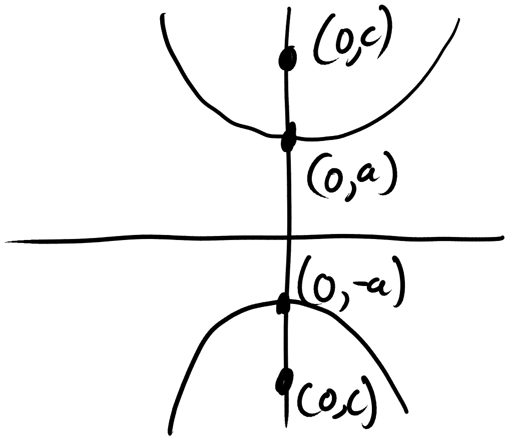
other branch $|PF_1| - |PF_2| = -2a$

Equation that includes both branches

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad c^2 = a^2 + b^2$$

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} > 0$$

swap x and y: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ $\quad c^2 = a^2 + b^2$



Which way does parabola open?

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{y^2}{a^2} = 1 + \frac{x^2}{b^2} > 0$$

Parameterise hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$x(t) = a \cosh t$$
$$y(t) = b \sinh t$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

$$\sinh t = \frac{e^t - e^{-t}}{2}$$