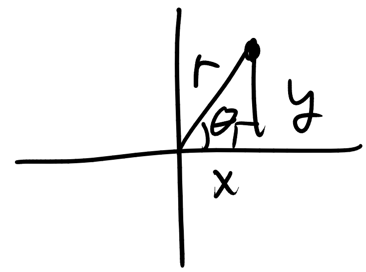
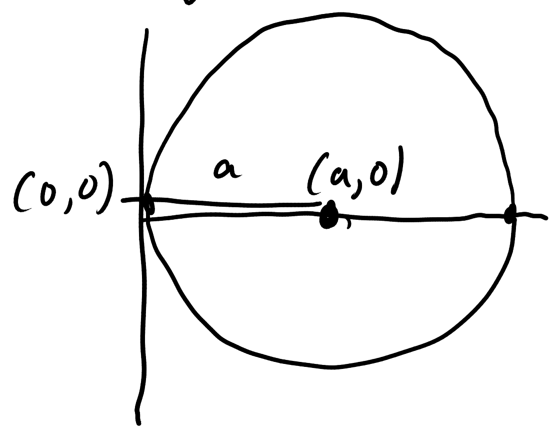
Polar coordinates & Calculus

Recall : $x = r\cos\theta$

$y = r\sin\theta$



What is polar equation of circle of radius a centered at $(a,0)$



$$(x-a)^2 + y^2 = a^2$$

$$(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$$

$$r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta = a^2$$

$$\left((A+B)^2 = A^2 + 2AB + B^2 \right)$$

a = radius of circle
r = distance from origin of a point on the circle

$$r^2(\cos^2\theta + \sin^2\theta) - 2ar\cos\theta + a^2 = a^2$$

$$r^2(1) \quad - 2ar\cos\theta = 0$$

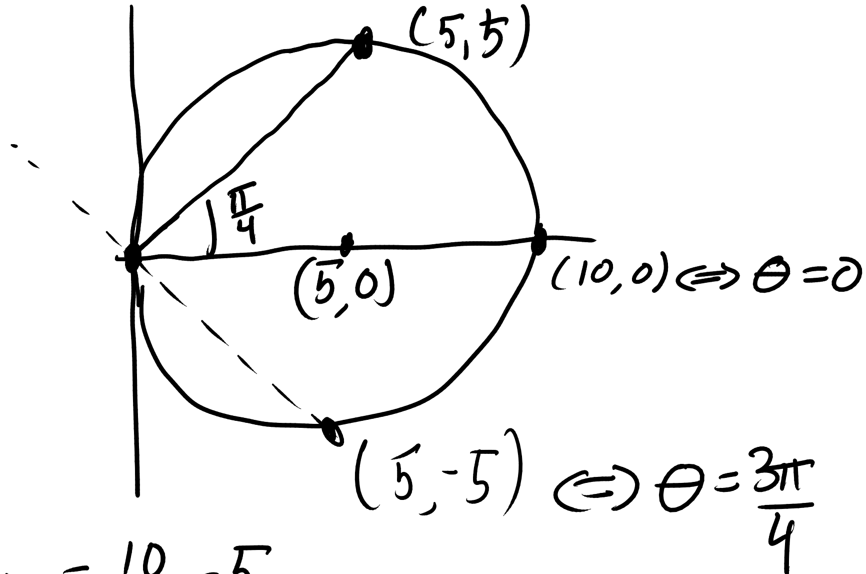
$$r^2 = 2ar\cos\theta$$

$$r = 2a\cos\theta$$

e.g. $\boxed{a = 5 \quad r = 10\cos\theta}$

$a = 5$

$r = 10\cos\theta$



- $\theta = \frac{\pi}{4}$ $\quad r = 10\cos\frac{\pi}{4}$

$$r = \frac{10}{\sqrt{2}}$$

$$x = r\cos\theta = \frac{10}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{10}{2} = 5$$

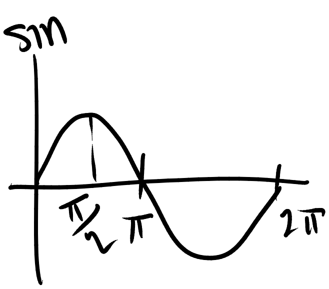
$$y = r\sin\theta = \frac{10}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{10}{2} = 5$$

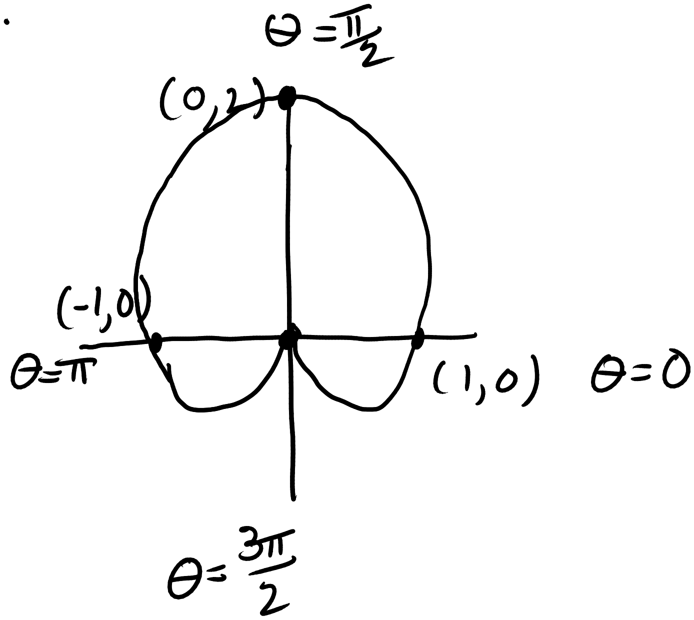
curve passes thru $(0,0)$ when $\theta = \frac{\pi}{2}$

$\theta = \frac{3\pi}{4}$ $\qquad r = 10\cos\frac{3\pi}{4} = -\frac{10}{\sqrt{2}}$

Covers whole circle once when θ goes from 0 to π.

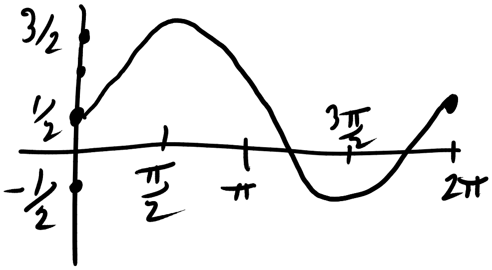
Ex: $r = 1 + \sin\theta$

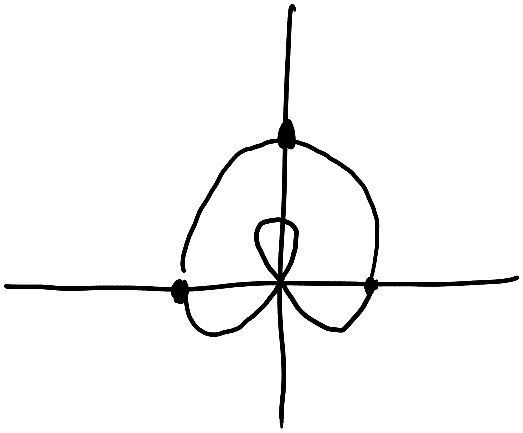




Cardioid

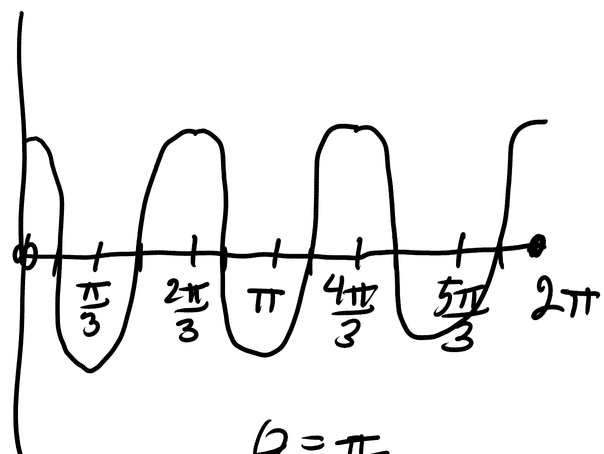
$r = \frac{1}{2} + \sin\theta$



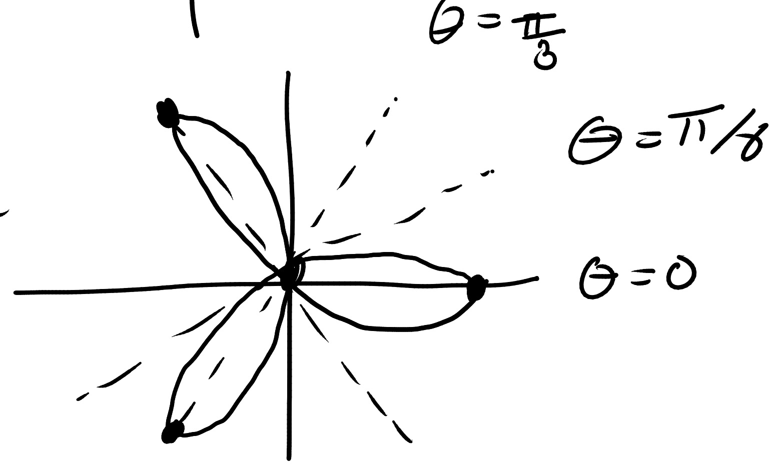


inner loop corresponds to values of θ where r is negative.

$r = \cos 3\theta$



3-leaved rose
trefoil



Covers curve once as θ goes from 0 to π
covers it again as θ goes from π to 2π.

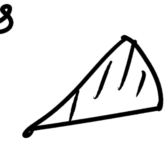
<u>Fun:</u> $r = \cos n\theta$

Calculus w/ polar curves :
- Tangent lines
- Arclength
- Areas.

Tangent lines and Arclength : combining cartesian formulas with $x = r\cos\theta$, $y = r\sin\theta$

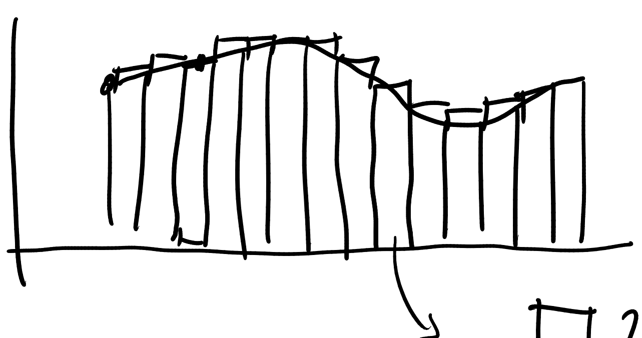
But today let's focus on Areas:

Cartesian Areas $\leftrightarrow$ based on rectangles
Polar Areas $\leftrightarrow$ based on sectors

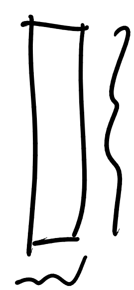


Cartesian integral

$$\int y\,dx \approx \sum_i y_i \,\Delta x_i$$



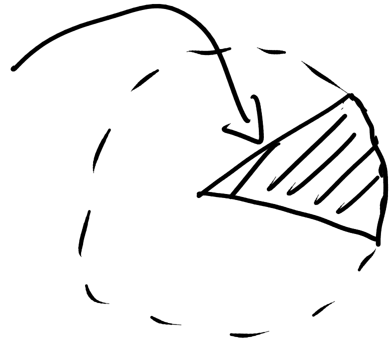
Approximate region by many rectangles

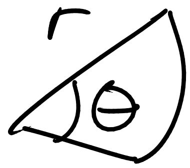


y_i $\quad A = y_i\,\Delta x_i$

Δx_i

Sector



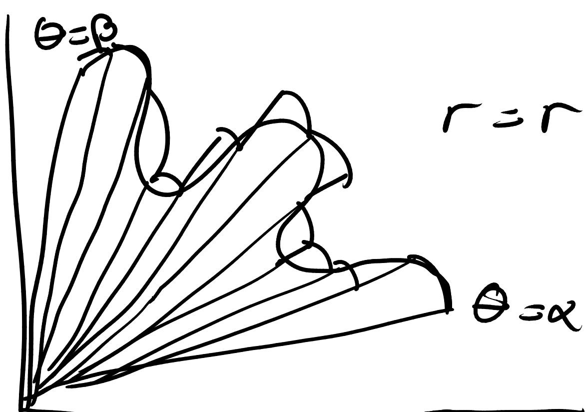


$$A = \left(\pi r^2\right)\left(\frac{\theta}{2\pi}\right)$$

Area of whole circle

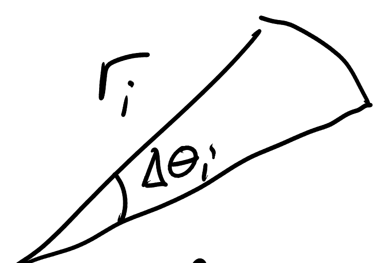
fraction of circle that you take

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$$



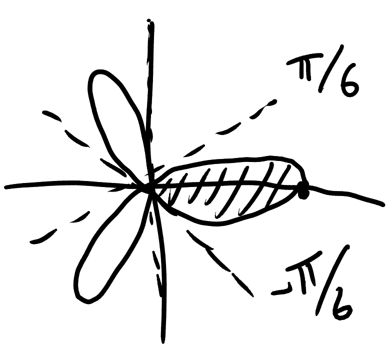
$r = r(\theta)$

Approximate by thin sectors



$$A = \sum_i \frac{1}{2} r_i^2 \Delta\theta_i \xrightarrow[\Delta\theta \to 0]{} \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta = A$$

$r = \cos 3\theta$ Area of one "leaf"?



$$A = \int_{-\frac{\pi}{6}}^{\pi/6} \frac{1}{2}\left[\cos 3\theta\right]^2 d\theta$$

$$A=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1}{2}\cos^2 3\theta\, d\theta$$

$\cos 2x=\cos^2 x-\sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x=\dfrac{1+\cos 2x}{2}$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$=\frac{1}{4}\left[\theta+\frac{1}{6}\sin 6\theta\right]_{-\frac{\pi}{6}}^{\pi/6}=\frac{1}{4}\left[\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right]$$

$$=\frac{1}{4}\,2\frac{\pi}{6}=\frac{\pi}{12}$$

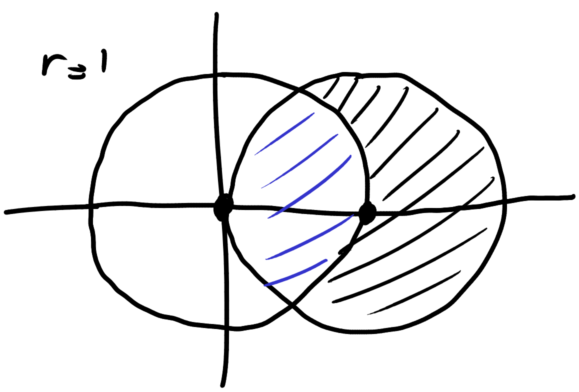
Finding area between two curves.

Find area inside $r=2\cos\theta$ but outside $r=1$.

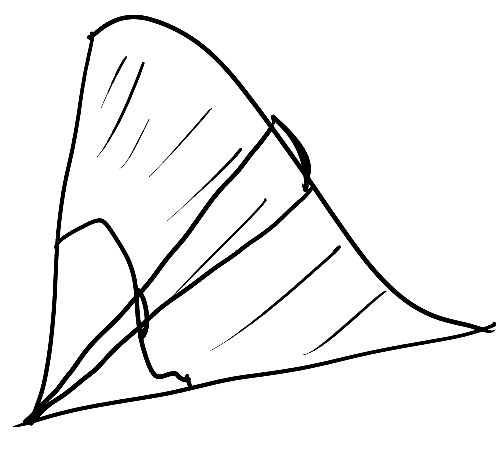
↑

radius 1

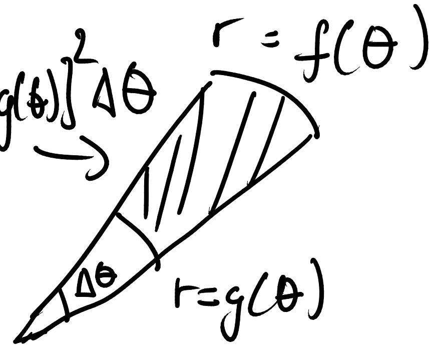
center $(1,0)$



Idea



$A=\frac{1}{2}f(\theta)^2\Delta\theta-\frac{1}{2}[g(\theta)]^2\Delta\theta$

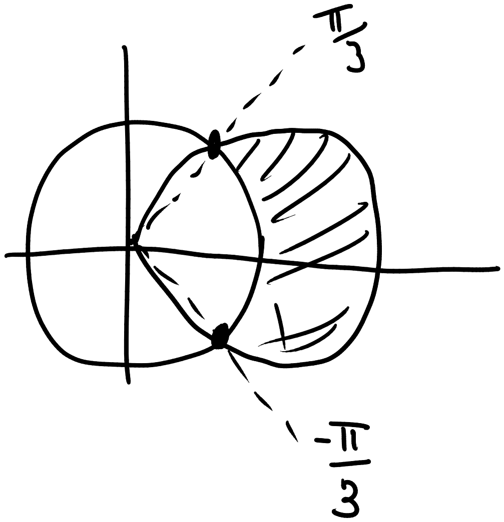


$$A=\frac{1}{2}[f(\theta)]^2\Delta\theta-\frac{1}{2}[g(\theta)]^2\Delta\theta=\frac{1}{2}\left([f(\theta)]^2-[g(\theta)]^2\right)\Delta\theta$$

Take $\Delta\theta \to 0$ $\quad A = \int \frac{1}{2}\left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta$

Area between two curves: $r = f(\theta)$ "outer"

$r = g(\theta)$ "inner"

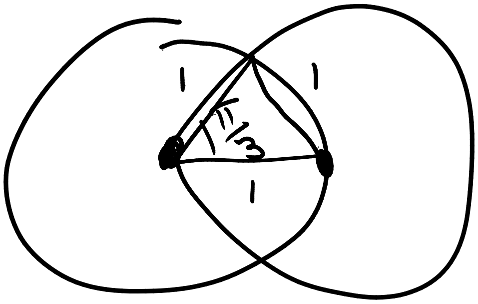


To find intersections

$$1 = r = 2\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\frac{\pi}{3}, \ -\frac{\pi}{3}$$



$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left([2\cos\theta]^2 - [1]^2\right) d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left(4\cos^2\theta - 1\right) d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left(4\left(\frac{1+\cos 2\theta}{2}\right) - 1\right) d\theta$$

$$= \frac{1}{2} \int_{-\frac{\pi}{3}}^{\pi/3} \left[1 + 2\cos 2\theta\right] d\theta$$

$$= \frac{1}{2}\left[\theta + \sin 2\theta\right]_{-\frac{\pi}{3}}^{\pi/3} = \frac{1}{2}\left[\frac{\pi}{3} + \sin\frac{2\pi}{3} - \left(-\frac{\pi}{3}\right) - \sin\frac{-2\pi}{3}\right]$$

$$= \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$