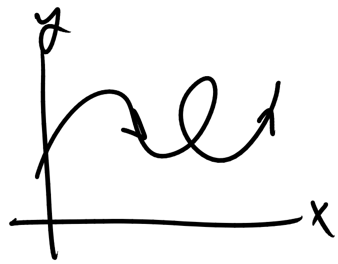
Area, Arclength, Polar coordinates

Recall last time



$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Slope of tangent line: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{y'(t)}{x'(t)}$

$x'(t)$ = horizontal velocity

$y'(t)$ = vertical velocity.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$\dfrac{d^2y}{dx^2}$?

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt}$$

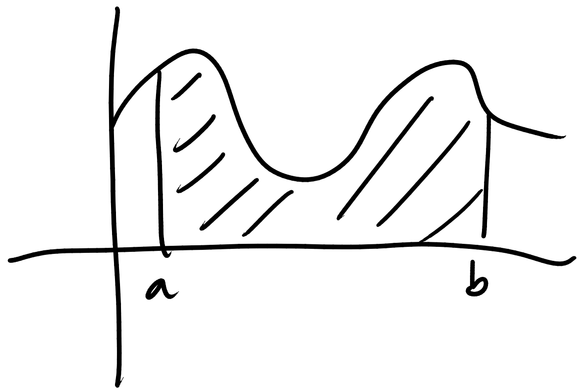
$$= \frac{d^2y}{dx^2} \cdot \frac{dx}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}$$

$$= \left[\frac{y'(t)}{x'(t)}\right]' \frac{1}{x'(t)}$$

<u>Area</u> under a curve defined parametrically

<u>Area</u> $y = f(x)$



$$\text{Area} = \int_a^b f(x)\,dx$$

$$= \int_a^b y\,dx \quad \leftarrow \text{can still use this for parametric equations}$$

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad dx = \frac{dx}{dt}\,dt = x'(t)\,dt$$

$$\text{Area} = \int_\alpha^\beta y(t)\,x'(t)\,dt$$

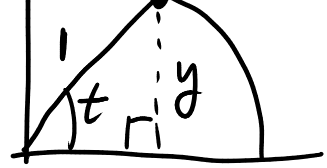
α = time corresponding to $x = a$

β = time corresponding to $x = b$

Same as u-substitution but "backward."

Ex area of a quarter-circle

radius 1



$$\begin{cases} x = \cos t \\ y = \sin t \\ 0 \le t \le \pi/2 \end{cases}$$

$$A = \int_{x=0}^{x=1} y\,dx = \int_{t=\frac{\pi}{2}}^{t=0} \sin t\,(-\sin t)\,dt$$

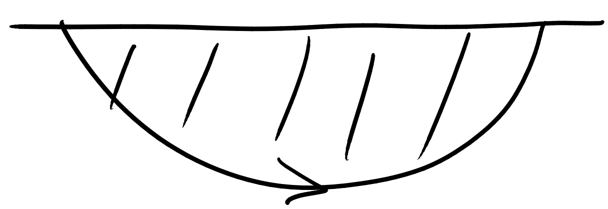
$$dx = -\sin t\,dt$$

$$= \int_{\frac{\pi}{2}}^{0} -\sin^2 t\,dt = -\int_{\frac{\pi}{2}}^{0} \sin^2 t\,dt$$

$$= \int_{0}^{\pi/2} \sin^2 t\,dt = \int_{0}^{\frac{\pi}{2}} \left(\frac{1-\cos 2t}{2}\right) dt$$

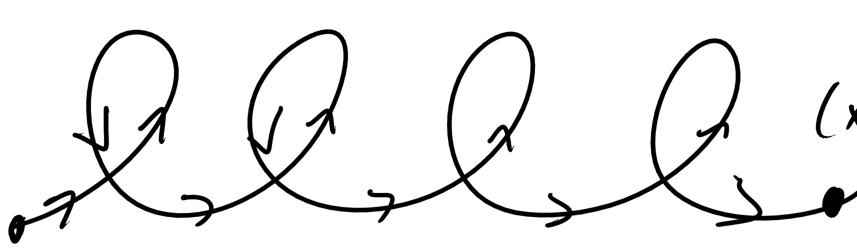
$$= \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_{0}^{\pi/2} = \frac{\pi}{4}$$

For Area: integrate from left to right.



$\int y\,dx$ will be negative.

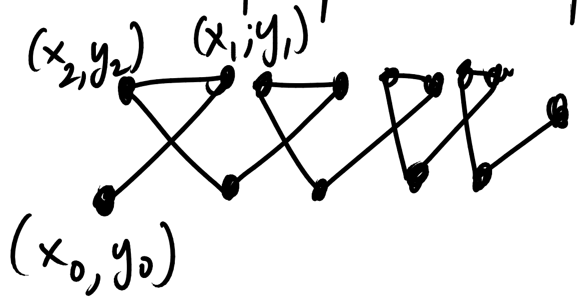
<u>Arclength</u>: more natural in parametric form:



$(x(t), y(t))$

How long is the path? How far do you travel?

What if path is piecewise straight?



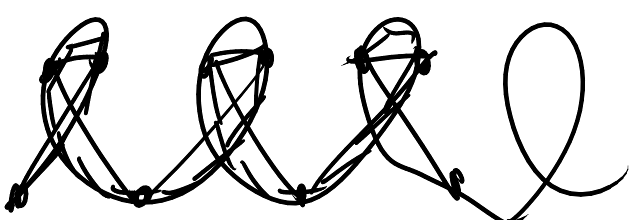
take length of each piece, add together

$$\text{length } (x_0,y_0) \rightarrow (x_1,y_1) = \sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$$

$$(x_1,y_1) \rightarrow (x_2,y_2) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$\vdots$$

$$\text{length} = \sum_{i=0}^{N} \sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}$$



$$L = \sum_{i=0}^{N} \sqrt{(\Delta x_i)^2+(\Delta y_i)^2} = \sum_{i=0}^{N} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2+\left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i$$

Take limit as $\Delta t_i \rightarrow 0$

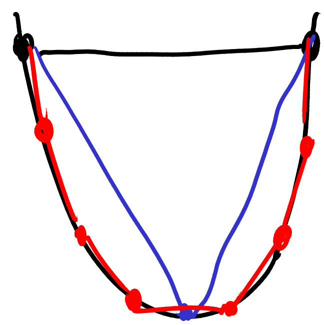
$$\frac{\Delta x}{\Delta t} \rightarrow \frac{dx}{dt} \qquad \frac{\Delta y}{\Delta t} \rightarrow \frac{dy}{dt}$$

$$\sum (\text{stuff})\,\Delta t \rightarrow \int (\text{stuff})\, dt$$

$$\sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t \to \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\text{Arc length} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Idea as $\Delta t \to 0$, the piecewise-straight approximations get closer to the actual curve.



Another way to understand

$x' = \frac{dx}{dt}$ is horizontal velocity

$y' = \frac{dy}{dt}$ is vertical velocity

$\vec{v} = \langle x'(t), y'(t) \rangle$

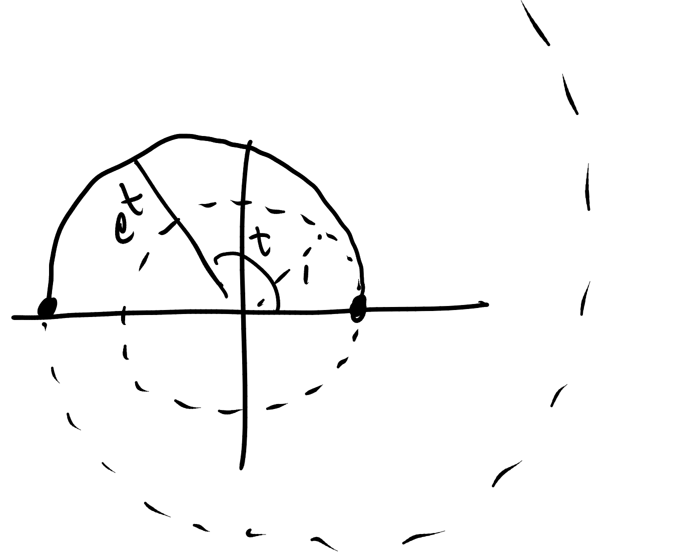
speed = magnitude of velocity $= \sqrt{(x')^2 + (y')^2}$

$(\text{speed})^2 = (x')^2 + (y')^2$

$$\text{Arclength} = \int (\text{speed})\, dt = \int \sqrt{(x')^2 + (y')^2}\, dt$$

"distance = rate x time"

Example $\begin{cases} x = e^t \cos t \\ y = e^t \sin t \\ 0 \le t \le \pi \end{cases}$



$e^0 = 1, \; e^\pi > 1$

$$x'(t) = e^t \cos t - e^t \sin t$$
$$y'(t) = e^t \sin t + e^t \cos t$$

$$(x'(t))^2 = (e^t \cos t)^2 - 2e^t \cdot e^t \cos t \sin t + (e^t \sin t)^2$$
$$(y'(t))^2 = (e^t \sin t)^2 + 2e^t \cdot e^t \sin t \cos t + (e^t \cos t)^2$$

add $= e^{2t} \cos^2 t + e^{2t} \sin^2 t + e^{2t} \sin^2 t + e^{2t} \cos^2 t$

$$= 2e^{2t}(\cos^2 t + \sin^2 t) = 2e^{2t}$$

$$\text{Arclength} = \int_0^\pi \sqrt{2e^{2t}}\, dt = \int_0^\pi \sqrt{2}\, e^t\, dt$$
$$= \sqrt{2}(e^\pi - 1)$$

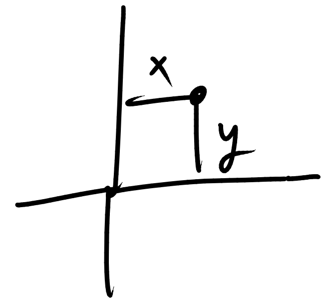
For Arclength only:

Always integrate from lower value of t to greater value of t

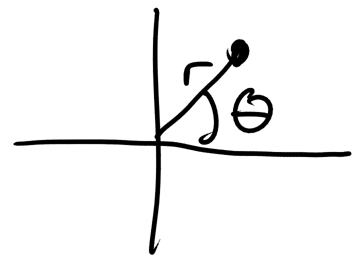
(Doesn't matter which direction curve is moving.)

Polar coordinates

Cartesian coordinates (x, y)

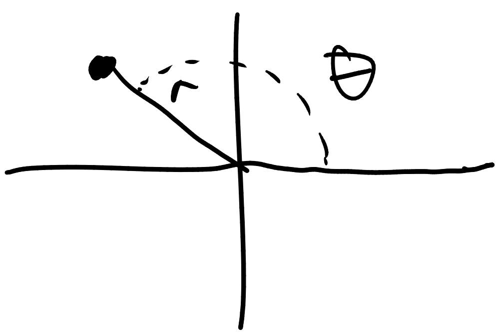


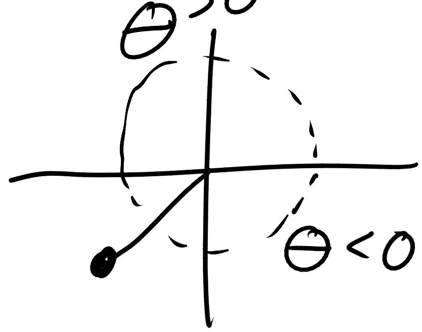
Polar coordinates (r, θ)



r = distance from origin

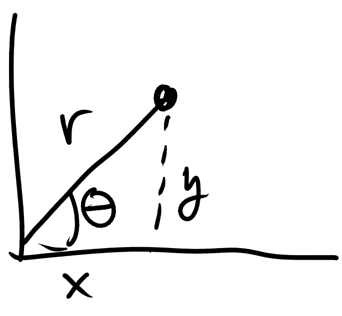
θ = angle from horizontal





multiple choices for θ

relating:



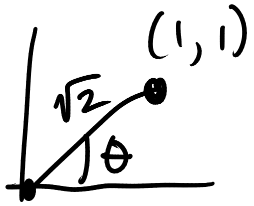
$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$ Fundamental

plug in (r, θ), get (x, y)

inverting: $x^2 + y^2 = r^2 \rightarrow r = \sqrt{x^2 + y^2}$

$$\tan\theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\frac{y}{x}$$

unless $x = 0$ slightly problematic

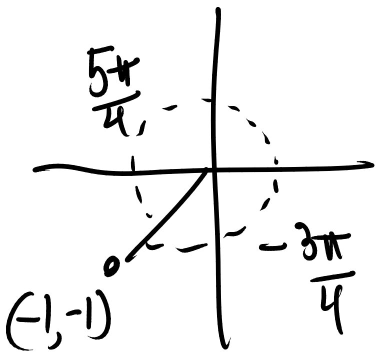


$r = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

$(r = \sqrt{2}, \theta = \frac{\pi}{4})$ is correct



$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\tan\theta = \frac{-1}{-1} = 1$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$??? wrong

But $\theta = \frac{5\pi}{4}$ or $\theta = -\frac{3\pi}{4}$ work

Reason: $\tan^{-1}(\_)$ always gives values $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

since tan is periodic with period π

$\tan(\theta - \pi) = \tan\theta = \tan(\theta + \pi) = \tan(\theta + 2\pi) \cdots$

When we have $\tan\theta = \frac{y}{x}$

we try to take $\theta = \tan^{-1}(\frac{y}{x})$ but the true answer may differ by a multiple of π

May have to think carefully

Other wierd things

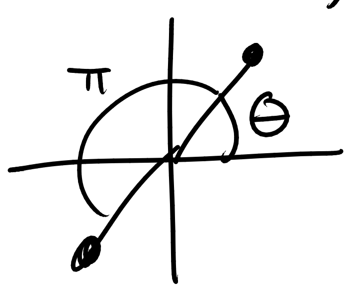
<u>(r,θ)</u>

$(0,\theta)$ = origin no matter what θ is.

(r,θ) and $(r,\theta+2\pi)$ are the same point.

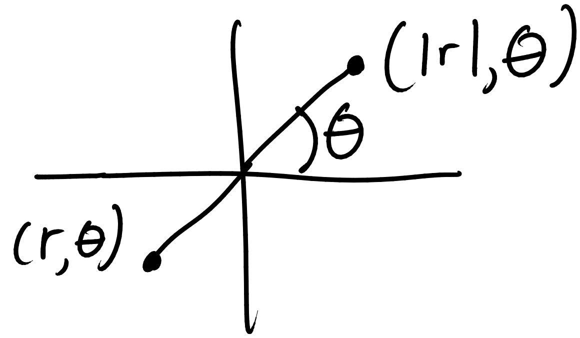
as are $(r,\theta+4\pi)$, $(r,\theta+6\pi)$, ...

· (r,θ) and $(r,\theta+\pi)$ are opposite



· Allowing r to be negative:

if $r<0$, (r,θ) is opposite to $(|r|,\theta)$



$r=-1$, $\theta=\frac{\pi}{4}$