408 M Multivariable calculus.

Single variable $y = f(x)$

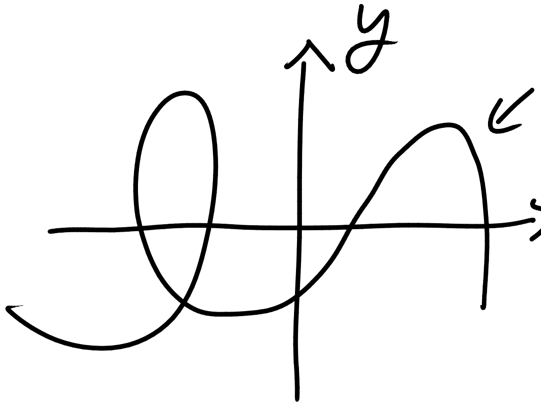
dependent variable, function, independent variable

Ways to increase complexity

- 1 independent, 2 dependent (Ch 10)
- 1 independent, 3 dependent (Ch 12+13)
- 2 or 3 independent variables, 1 dependent (Ch 14+15)
- More variables make the subject more geometrically complicated.
- Another slogan "one variable at a time" (sometimes)

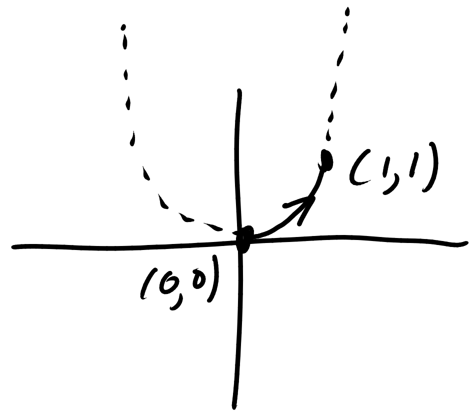
Parametric Curves: $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

t independent variable "time"
x, y dependent variables "coordinates of position"

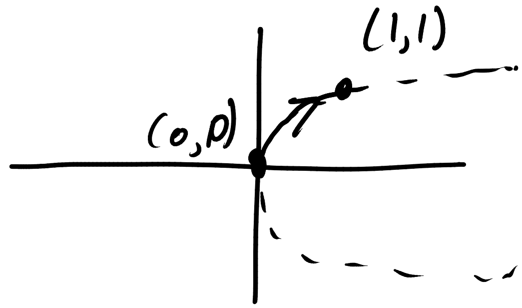


plot of $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

$\begin{cases} x = t \\ y = t^2 \\ 0 \le t \le 1 \end{cases}$ $\rightarrow y = x^2$

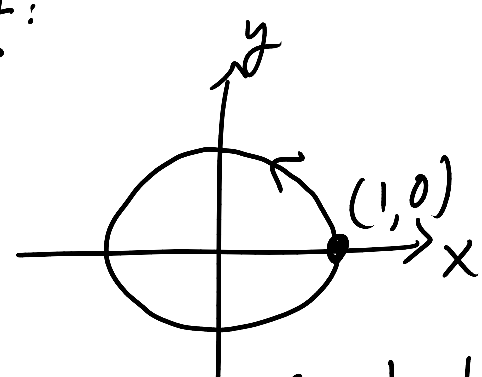


$\begin{cases} x = t^2 \\ y = t \quad (x = y^2) \\ 0 \le t \le 1 \end{cases}$



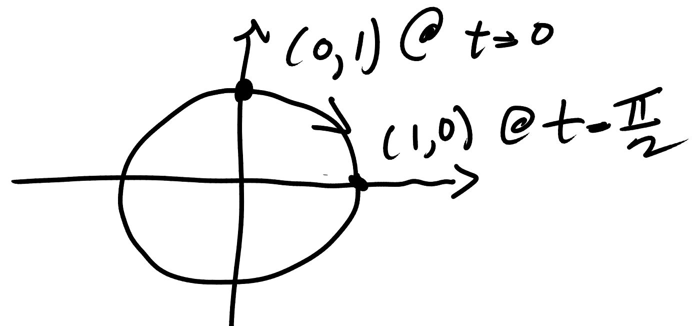
Example to <u>know by heart</u>:

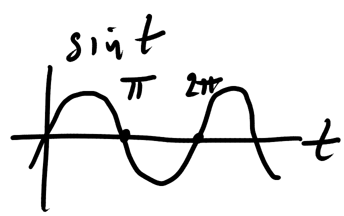
$\begin{cases} x = \cos t \\ y = \sin t \\ 0 \le t \le 2\pi \end{cases}$



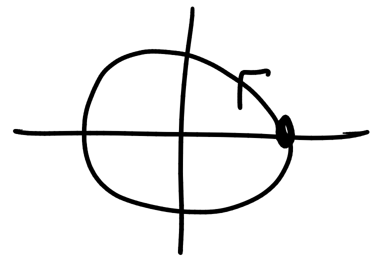
counterclockwise

$\begin{cases} x = \sin t \\ y = \cos t \\ 0 \le t \le 2\pi \end{cases}$





$$\begin{cases} x = \cos 2t \\ y = \sin 2t \\ \quad 0 \le t \le 2\pi \end{cases}$$



"wraps around twice"

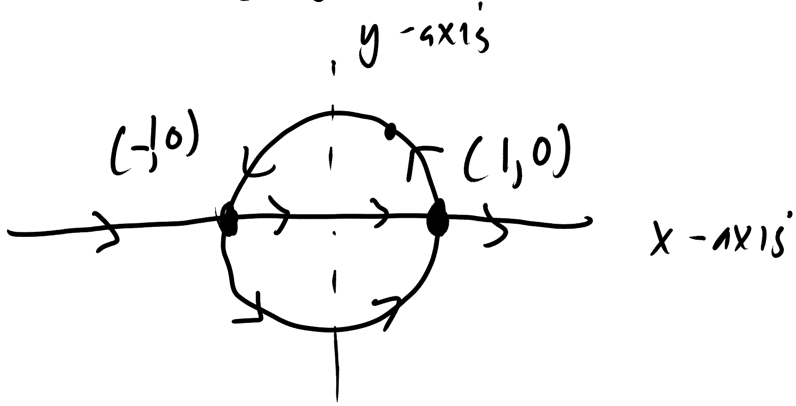
the parametric equations have more information than the plot by itself / "geometric image"

→ often makes it easier to work with than the cartesian equation $(x^2 + y^2 = 1)$

<u>Example</u>: "intersections" vs "collisions"

Particle 1: $\begin{cases} x_1(t) = t \\ y_1(t) = 0 \end{cases}$

Particle 2: $\begin{cases} x_2(t) = \cos t \\ y_2(t) = \sin t \end{cases}$



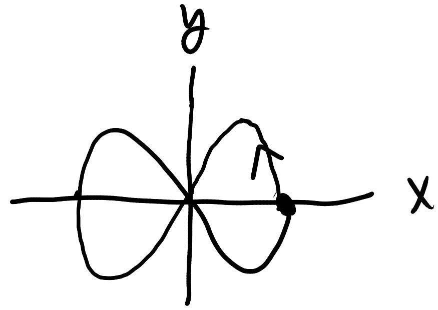
intersections:
$(-1,0)$, $(1,0)$

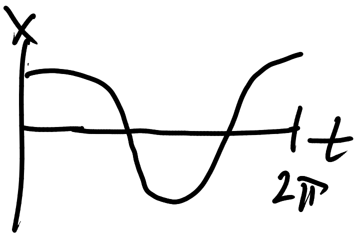
Collision means in the same place at same time:

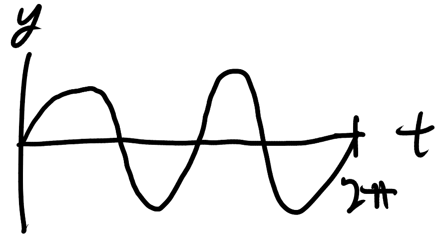
At $(1,0)$: particle 1 gets there at $t = 1$

but at $t = 1$, particle 2 is at $(\cos 1, \sin 1)$
$= (.54.., .84..)$

$$\begin{cases} x = \cos t \\ y = \sin 2t \\ 0 \le t \le 2\pi \end{cases}$$







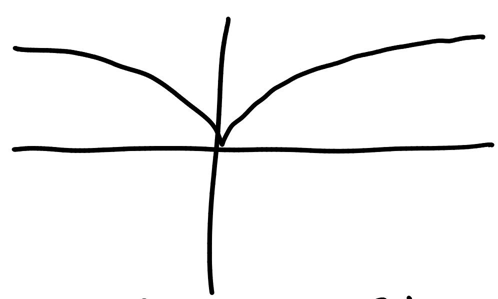
<u>Fun</u> : $\begin{cases} x = \cos t \\ y = \sin nt \\ 0 \le t \le 2\pi \end{cases}$ n is an integer $n = 1, 2, 3, 4, 5, \ldots$

Eliminating parameter

$x = t$, $y = t^2 \Rightarrow y = x^2$

$x = t^3$, $y = t^2 \rightarrow$ $t = x^{1/3}$ $\Rightarrow$ $y = x^{2/3}$

$y^3 = x^2$



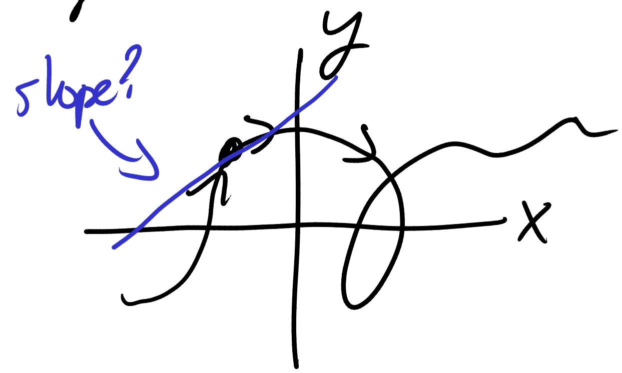
$x = \cos t$, $y = \sin t$

$x^2 = \cos^2 t$, $y^2 = \sin^2 t$ $\Rightarrow$ $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

Derivative / slope of tangent line.

Given $\begin{cases} x(t) \\ y(t) \end{cases}$



Want slope $= \dfrac{dy}{dx}$

Know $\begin{cases} x(t) \\ y(t) \end{cases} \implies \begin{aligned} x'(t) &= \dfrac{dx}{dt} \\ y'(t) &= \dfrac{dy}{dt} \end{aligned}$

$y(t) = f(x(t))$ some f that theoretically exists

$y'(t) = f'(x(t))\, x'(t)$

$\dfrac{dy}{dx} = f'(x(t)) = \dfrac{y'(t)}{x'(t)}$

$\dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{y'(t)}{x'(t)}$

Mnemonic
$\dfrac{dy}{dt} = \dfrac{dy}{\cancel{dx}} \dfrac{\cancel{dx}}{dt}$

Physically $x'(t)$ and $y'(t)$ are the velocities of the particle

Find $\frac{dy}{dx}$ = slope of tangent line.

$$x = t\sin t \qquad \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$
$$y = t^2 + t$$

$$x'(t) = t\cos t + \sin t$$
$$y'(t) = 2t+1$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t+1}{t\cos t + \sin t}$$ slope as a function of t

What is the equation of tangent line at $t = \frac{\pi}{2}$?

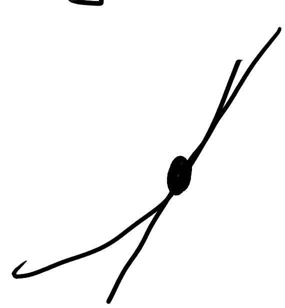
$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{2\left(\frac{\pi}{2}\right)+1}{\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}} = \frac{\pi+1}{1} = \pi+1$$

position at $t = \frac{\pi}{2}$

$$x = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$
$$y = \left(\frac{\pi}{2}\right)^2 + \frac{\pi}{2}$$

At point $\left(\frac{\pi}{2}, \left(\frac{\pi}{2}\right)^2 + \frac{\pi}{2}\right)$ slope $= \pi+1$

$$(y - y_0) = m\,(x - x_0)$$
$$y - \left[\left(\frac{\pi}{2}\right)^2 + \frac{\pi}{2}\right] = (\pi+1)\left(x - \frac{\pi}{2}\right)$$



same question at $t=0$

$x = t\sin t$ $\qquad \dfrac{dy}{dx} = \dfrac{2t+1}{t\cos t + \sin t}$

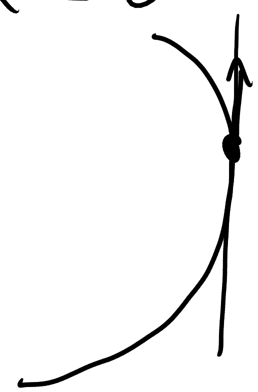
$y = t^2 + t$

$= \dfrac{2(0)+1}{0\cos 0 + \sin 0} = \dfrac{1}{0} \to$ vertical

$\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$ when $x'(t) = 0$, we're dividing by 0, and the tangent line is vertical.

Physically x', y' represent velocity

$x' = 0$ means zero horizontal velocity



particle is moving vertically.

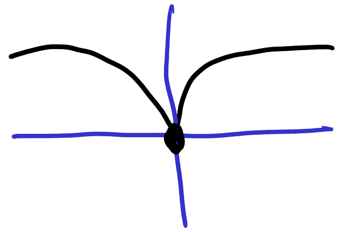
on the other hand, if $y' = 0$, horizontal tangent line, zero vertical velocity

Special what if both $x' = 0$, $y' = 0$?
there may be no well-defined tangent line

cusp:

$x = t^3$

$y = t^2$



Tangent line does not exist at $(0,0)$