3.4

69 a) $F'(x) = f'(e^x)(e^x)' = f'(e^x)e^x$

b) $G'(x) = e^{f(x)} f'(x)$.

76 $y' = re^{rx}$ $y'' = r^2 e^{rx}$

$y'' - 4y' + y = e^{rx}(r^2 - 4r + 1)$

When $r^2 - 4r + 1 = 0$, $y'' - 4y' + y = 0$.

$\Rightarrow r = 2 \pm \sqrt{3}$.

77 $y^{(50)} = -2^{50}\cos(2x)$.

97. $\frac{dy}{du} = f'(u)$ $\frac{d^2y}{du^2} = f''(u)$

$\frac{du}{dx} = g'(x)$ $\frac{d^2u}{dx^2} = g''(x)$.

Thus right-hand-side $= f''(u)(g'(x))^2 + f'(u)g''(x)$

$\frac{dy}{dx} = f'(u)\frac{du}{dx} = f'(u)g'(x)$.

$\frac{d^2y}{dx^2} = f''(u)g'(x)g'(x) + f'(u)g''(x) = f''(u)(g'(x))^2 + f'(u)g''(x)$

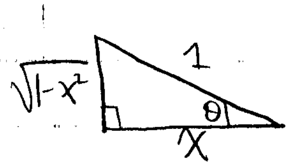
$=$ Right-hand side

3.5

63. Define $f(x) = \cos^{-1}(x)$ $g(y) = \cos(y) \Rightarrow g \circ f(x) = x$.

Chain Rule: $g'(f(x))f'(x) = 1$

$\Rightarrow f'(x) = 1/g'(f(x)) = 1/-\sin(\cos^{-1}x) = -\frac{1}{\sin(\cos^{-1}x)}$ (*)



$\theta = \cos^{-1}x$. $\sin(\cos^{-1}x) = \sin\theta = \sqrt{1-x^2}$

Plug in (*): $f'(x) = -\frac{1}{\sqrt{1-x^2}}$.

77. a) $f \circ f^{-1}(x) = x$

$\Rightarrow f'(f^{-1}(x))(f^{-1})'(x) = 1 \Rightarrow (f^{-1})'(x) = 1/f'(f^{-1}(x))$

b). $(f^{-1})'(5) = 1/f'(f^{-1}(5)) = 1/f'(4) = 1/\frac{2}{3} = 3/2$.

3.6

11 $g'(x) = \frac{1}{x\sqrt{x^2-1}}\left(\sqrt{x^2-1} + x\frac{1}{2}(x^2-1)^{-\frac{1}{2}}\cdot 2x\right)$.

$= \frac{1}{x\sqrt{x^2-1}}\left(\sqrt{x^2-1} + x^2/\sqrt{x^2-1}\right)$

20. $H(z) = \frac{1}{2}\ln\left(\frac{a^2-z^2}{a^2+z^2}\right) = \frac{1}{2}\left(\ln(a^2-z^2) - \ln(a^2+z^2)\right)$

$H'(z) = \frac{1}{2}\left(\frac{1}{a^2-z^2}(-2z) - \frac{1}{a^2+z^2}(2z)\right)$

$= -\frac{z}{a^2-z^2} - \frac{z}{a^2+z^2}$

42. $\ln y = \ln(\sqrt{x}) + \ln(e^{x^2-x}) + \ln(x+1)^{2/3}$

$= \frac{1}{2}\ln x + x^2 - x + \frac{2}{3}\ln(x+1)$

$\frac{d}{dx}\ln y = \frac{1}{2}\frac{1}{x} + 2x - 1 + \frac{2}{3}\frac{1}{x+1}$

Meanwhile $\frac{d}{dx}\ln y = \frac{1}{y}y'$

Thus

$y' = y\left(\frac{d}{dx}\ln y\right) = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3}\left(\frac{1}{2x} + 2x + \frac{2}{3}\frac{1}{x+1} - 1\right)$.

45. $\ln y = \sin x\ln x \Rightarrow \frac{d}{dx}(\ln y) = \cos x\ln x + \frac{\sin x}{x}$

$\Rightarrow y' = y\left(\frac{d}{dx}\ln y\right) = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)$.

3.7

10 a) $S' = 4t^3 - 12t^2 - 40t + 20$

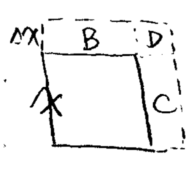
$S'(t) = 20 \Rightarrow t_1 = 0$ s $t_2 = 5$ s

b) $S''(t) = 12t^2 - 24t - 40$

$S''(t) = 0 \Rightarrow t = 1 + \frac{\sqrt{39}}{3}$ s

11. a) $A(x) = x^2$. $A'(x) = 2x$. $A'(15) = 30$.

b) $A'(x) = 2x = \frac{1}{2}\cdot 4x = \frac{1}{2}$ perimeter.



ΔA = Area of B + Area of C + Area of D

$= \Delta x\cdot x + \Delta x\cdot x + (\Delta x)^2 = \Delta x(2x) + (\Delta x)^2$.

When Δx is small, $(\Delta x)^2 \simeq 0$. $\Delta A \simeq 2x\Delta x$ $\frac{\Delta A}{\Delta x} \simeq 2x = \frac{1}{2}$ perimeter

3.9

27\. Let h be the height of the pile, and V be its volume.

$V(h) = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

Differentiate with respect to t

$\frac{d}{dt}V = \frac{1}{4}\pi h^2 \frac{d}{dt}h$

Plug in $\frac{d}{dt}V = 30$ $h = 10$ $\Rightarrow \frac{d}{dt}h = \frac{6}{5\pi}$ ft/min

33\. $V = C/P \Rightarrow \frac{d}{dt}V = C\left(-\frac{1}{P^2}\right)\frac{d}{dt}P = -\frac{C}{150^2}\cdot 20$

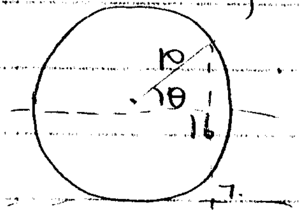
Meanwhile,

$C = 600 \times 150$

So $\frac{d}{dt}V = -\frac{600\times150}{150^2}\cdot 20 = -80$ kPa/min

42

0.5r/min



$h(\theta) = 10 + 10\sin\theta$.

$\frac{d}{dt}h = 10\cos\theta\frac{d\theta}{dt}$

When $h = 16$, $\sin\theta = 6/10 = 3/5$, $\cos\theta = 4/5$.

$\frac{d\theta}{dt} = 0.5\text{r/min} = \pi$ /min

$\frac{d}{dt}h = 8\pi$ m/min

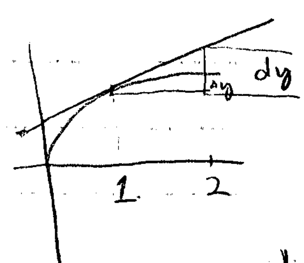
3.10

27\. $\tan 44^\circ - \tan 45^\circ \simeq \tan' 45^\circ(-1) = -\frac{\pi}{90}$

$\Rightarrow \tan 44^\circ \simeq 1 - \pi/90 \simeq 0.965$

20\. $\Delta y = \sqrt{1+1} - \sqrt{1} = \sqrt{2} - 1$

$dy = \frac{1}{2}x^{-\frac{1}{2}}\Big|_{x=1}\cdot 1 = \frac{1}{2}$.



36 $\frac{1}{2}\cdot\frac{4}{3}\pi\left(\left(\frac{50}{2} + 5\times10^{-4}\right)^3 - \left(\frac{50}{2}\right)^3\right) \simeq \frac{2}{3}\pi\ 3x^2\Big|_{x=\frac{50}{2}}\cdot(5\times10^{-4})\ m^3$

$\simeq 1.96\ m^3$

4.1

63. $f'(x)=ax^{a-1}(1-x)^b+x^a(-b)(1-x)^{b-1}$

$=x^{a-1}(1-x)^{b-1}(a(1-x)-bx)$

$=x^{a-1}(1-x)^{b-1}(-ax-bx+a)$

$f'(x)=0 \Rightarrow x=\frac{a}{a+b}$

$f\left(\frac{a}{a+b}\right)=\left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b$

$f(0)=f(1)=0$

$\Rightarrow \max f(x)=\left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b$

70. $F'(\theta)=\mu W\left(-\frac{1}{(\mu\sin\theta+\cos\theta)^2}\right)(\mu\cos\theta-\sin\theta)$

$=\mu W\left(-\frac{\cos\theta}{(\mu\sin\theta+\cos\theta)^2}\right)(\mu-\tan\theta)$

Since $\mu W\left(-\frac{\cos\theta}{(\mu\sin\theta+\cos\theta)^2}\right)\leq 0$, $F'(\theta)\geq 0$ when $\tan\theta\geq\mu$ that is when $\theta>\arctan\mu$. $F'(\theta)\leq 0$ when $\tan\theta<\mu$, that is, when $\theta<\arctan\mu$.

So $\min F(\theta)=F(\arctan\mu)$

74. $g'(x)=3(x-5)^2 \Rightarrow g'(5)=0$ 5 is a critical number.

For any positive number δ, $g(5+\delta)-g(5)=\delta^3>0$

$g(5-\delta)-g(5)=-\delta^3<0$

So in any small region containing 5, there are always x' and x'' where $g(x')>g(5)$ and $g(x'')<g(5)$.

Thus no local extreme value at 5.

4.2

5. $f(-1)=f(1)=0$

$f'(x)=-\frac{2}{3}x^{-1/3}\neq 0$ for all c.

Because f not differentiable at 0

6. $f(0)=f(\pi)=0$

$f'(x) = \sec^2 x \neq 0$ on $(0, \pi)$.

Because f is not differentiable at $x = \pi/2$.

15. $\frac{f(4) - f(1)}{4-1} = \frac{1 - 2^{-2}}{3} = \frac{1}{4}$

$f'(x) = -2(x-3)^{-3} \neq \frac{1}{4}$ on $(1, 4)$.

f not differentiable at 3.

24. $f(8) - f(2) = f'(c)(8-2) = 6f'(c)$

$\Rightarrow 18 \leq f(8) - f(2) \leq 30$.

27. $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$

$f'(x) = \frac{1}{2} - \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2}\frac{\sqrt{1+x}-1}{\sqrt{1+x}}$

$f'(x) = 0 \Rightarrow x = 0$. For $x > 0$, $f'(x) > 0 \Rightarrow f$ increasing on $(0, +\infty)$

$f(x) \geq f(0) = 0$.

4.3

44 $S'(x) = 1 - \cos x$. $S'(x) = 0 \Rightarrow x = 0$, 2π and 4π.

$S'(x) \geq 0 \Rightarrow S$ always increasing.

$\min S = S(0) = 0$ $\max S = S(4\pi) = 4\pi$.

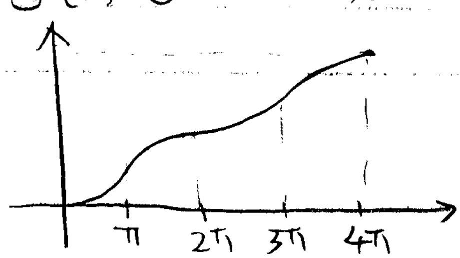
$S''(x) = \sin x$ $S''(x) \geq 0 \Leftrightarrow x$ in $[0, \pi] \cup [2\pi, 3\pi]$.

$S''(x) \leq 0 \Leftrightarrow x$ in $[\pi, 2\pi] \cup [3\pi, 4\pi]$.

Concave up on $[0, \pi] \cup [2\pi, 3\pi]$

Concave down on $[\pi, 2\pi] \cup [3\pi, 4\pi]$.

$S''(x) = 0 \Rightarrow x = 0, \pi, 3\pi, 4\pi$. These are inflection pts.



4.4

77 $\lim_{n\to\infty}(1+\frac{r}{n})^{nt} = \lim_{n\to\infty} e^{tn\ln(1+\frac{r}{n})}$

$\lim_{n\to\infty} n\ln(1+\frac{r}{n}) = \lim_{n\to\infty}\frac{\ln(1+\frac{r}{n})}{1/n} = \lim_{x\to 0}\frac{\ln(1+rx)}{x} \overset{\text{L'Hopital}}{=} \lim_{x\to 0}\frac{r}{1+rx} = r.$

Since $f(x)=e^x$ is continuous.

$\lim_{n\to\infty} e^{tn\ln(1+r/n)} = e^{rt}$,

$A = A_0 e^{rt}$.

81. $\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}} = \lim_{x\to a}\frac{\frac{1}{2}(2a^3x-x^4)^{-\frac{1}{2}}(2a^3-4x^3)-a^{\frac{5}{3}}\frac{1}{3}x^{-\frac{2}{3}}}{-a^{\frac{1}{4}}\frac{3}{4}x^{-1/4}}$

$= \frac{16}{9}a$

86. $\lim_{x\to 0}\left(\frac{\sin 2x+ax^3+bx}{x^3}\right) = \lim_{x\to 0}\frac{2\cos 2x+3ax^2+b}{3x^2}$

If $b\neq -2$, $\lim_{x\to 0}\frac{2\cos 2x+3ax^2+b}{3x^2}$ does not exist since $\lim_{x\to 0}3x^2=0$ while $\lim_{x\to 0}(2\cos 2x+3ax^2+b) = 2+b\neq 0$

If $b=-2$, $\lim_{x\to 0}\frac{2\cos 2x+3ax^2+b}{3x^2} = \lim_{x\to 0}\frac{-4\sin 2x+6ax}{6x}$

$= \lim_{x\to 0}\frac{-8\cos 2x+6a}{6}$

$= \frac{-8+6a}{6} = 0.$

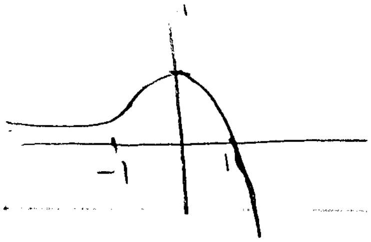
Thus $a=\frac{4}{3}$ $b=-2$

4.5

42. $f'(x) = -xe^x$. $x>0 \Rightarrow f'(x)<0$ $x<0 \Rightarrow f'(x)>0$.

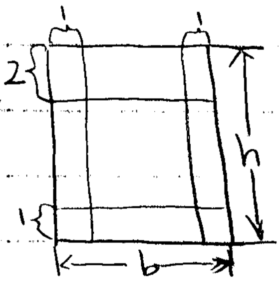
$f''(x) = -(1+x)e^x$ $x>-1$ concave down $x<-1$ concave up

$f(0)=1$ $f(-1)=2e^{-1}$ $f(1)=0$.



4.7

34.



$$A(b) = (h-3)(b-2)$$
$$= \left(\frac{180}{b} - 3\right)(b-2)$$
$$= 180 - 3b - \frac{360}{b} + 6$$
$$= 186 - 3\left(b + {}^{120}\!/_{b}\right)$$

$$A'(b) = -3\left(1 + \frac{120}{-b^2}\right) = 3\left(\frac{120 - b^2}{b^2}\right).$$

$A'(b) = 0 \Rightarrow b = 2\sqrt{30}$

If $b < 2\sqrt{30}$, $A'(b) > 0$. If $b > 2\sqrt{30}$, $A'(b) < 0$.

So $\max A(b) = A(2\sqrt{30})$.

Dimension: $2\sqrt{30} \times 3\sqrt{30}$

42. Same as Problem 70 in Sec. 4.1.