Solutions to ungraded homework: Lectures 17–25

M 408C, University of Texas at Austin

December 6, 2013

Lecture 17: Section 4.9

49. The statement that the slope of the tangent line at $(x, f(x))$ is $2x+1$ means $f'(x) = 2x+1$. Thus $f(x)$ is an antiderivative of this expression, and we conclude $f(x) = x^2 + x + C$ for some constant C. The statement that the graph of f passes through $(1,6)$ means $f(1) = 6$, or

$$f(1) = (1)^2 + 1 + C = 6,$$

so $C = 4$. Thus $f(x) = x^2 + x + 4$. We then find $f(2) = 10$.

62. Given data: acceleration $a(t) = 3\cos t - 2\sin t$, initial position $s(0) = 0$, initial velocity $v(0) = 4$. The velocity $v(t)$ is an antiderivative fo $a(t)$. Using known antiderivatives for $\sin t$ and $\cos t$ we get

$$v(t) = 3\sin t + 2\cos t + C.$$

Now $v(0) = 2 + C$, and since we must have $v(0) = 4$ we take $C = 2$. The position $s(t)$ is an antiderivative of $v(t)$, so we get

$$s(t) = -3\cos t + 2\sin t + 2t + D.$$

Now $s(0) = -3 + D$, and since we must have $s(0) = 0$, we have $D = 3$. The result is

$$s(t) = -3\cos t + 2\sin t + 2t + 3.$$

67. The object is thrown upward with initial velocity v_0 and initial position s_0. The units are meters and seconds. We are to show

$$[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0].$$

To prove this equation, we check that it holds at $t = 0$, and that the derivative of the equation is valid at all times. When $t = 0$, the equation

$$[v(0)]^2 = v_0^2 - 19.6[s(0) - s_0]$$

is valid because $s(0) = s_0$ and $v(0) = v_0$, and thus the expression $[s(0) - s_0]$ is zero, and $[v(0)]^2 = v_0^2$. Now take the derivative with respect to t of the equation we are to prove. This becomes

$$2v(t)v'(t) = -19.6s'(t)$$

because all of the constant terms go away. Because $s'(t) = v(t)$ by definition, we must prove $2v'(t) = -19.6$, that is $v'(t) = -9.8$. But $v'(t) = a(t)$ is the acceleration; since the object is subject to gravity, $a(t) = -9.8\ m/s^2$, so the equation holds.

A comment about the method of proof. Another way to look at it is that we consider the function $f(t) = [v(t)]^2 - v_0^2 + 19.6[s(t) - s_0]$ given by the difference of the two sides of the equation we wish to prove. What we have shown is that $f(0) = 0$ and $f'(t) = 0$. Since a function with zero derivative is constant, we find $f(t) = 0$ for all t, meaning the desired equation is valid for all t.

Lecture 18: Section 5.2

30. Express $\int_1^{10} (x - 4\ln x)\,dx$ as a limit of Riemann sums. For n subdivisions, the width of each subinterval is $\Delta x = (10 - 1)/n = 9/n$. Let us sample at the right endpoints, namely at the points $x_i = 1 + i(9/n)$ for i in the range 1 to n. The Riemann sum is then

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left[\left(1 + \frac{9i}{n}\right) - 4\ln\left(1 + \frac{9i}{n}\right) \right] \frac{9}{n}$$

The integral is then the limit as n goes to ∞

$$\int_1^{10} (x - 4\ln x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{9i}{n}\right) - 4\ln\left(1 + \frac{9i}{n}\right) \right] \frac{9}{n}$$

52. We have $F(x) = \int_2^x f(t)\,dt$, where the graph of $f(t)$ is given, and $f(t)$ is positive between 0 and 2, and negative between 2 and 5. The greatest value of the function $F(x)$ is $F(2) = \int_2^2 f(t)\,dt = 0$. To justify this, we claim that $F(x)$ is negative for other values of x. If x is between 2 and 5, then $F(x)$ is the integral of a negative function, and so it must be negative. If x is between 0 and 2, we are integrating a positive function, but the limits of integration are in the wrong order—the upper limit is less than the lower limit—so the integral is once again negative.

72. Express as an integral the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + (i/n)^2}.$$

To recognize this as a Riemann sum, we see that we are sampling the function $f(x) = \frac{1}{1+x^2}$ at the points $x_i = i/n$. These points are separated by the interval $\Delta x = \frac{1}{n}$, which we also recognize as the factor in front of the sum. The overall interval has length $n\Delta x = 1$, and because $x_n = 1$ for all n, we reckon that we must be integrating from 0 to 1. Thus

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2} = \int_0^1 \frac{1}{1+x^2}\,dx$$

Lecture 20: Section 5.4

4. Verify by differentiation that

$$\int \frac{x}{\sqrt{a+bx}}\,dx = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx} + C$$

Differentate the right-hand side. The constant goes away, and the product rule yields

$$\frac{2}{3b^2}\left[b\sqrt{a+bx} + (bx-2a)\frac{1}{2\sqrt{a+bx}}b\right]$$

Let us factor $\frac{b}{\sqrt{a+bx}}$ out of the brackets

$$\frac{2}{3b^2}\left[\frac{b}{\sqrt{a+bx}}\right][a+bx+(bx-2a)/2]$$

Simplify:

$$\frac{2}{3b^2}\left[\frac{b}{\sqrt{a+bx}}\right][3bx/2] = \frac{x}{\sqrt{a+bx}}$$

And we're done.

53. If oil leaks from a tank at a rank of $r(t)$ gallons per minute, then $\int_0^{120} r(t)\,dt$ is the total amount of oil, in gallons, that leaks from the tank in the two hour period from $t = 0$ to $t = 120$.

54. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. Then $100 + \int_0^{15} n'(t)\,dt$ represents the honeybee population 15 weeks after the start.

Lecture 22: Sections 6.2 and 7.1

61. We slice the torus horizontally into washers. The cross-section of the torus is a circle whose equation is $(x-R)^2 + y^2 = r^2$. Using y as the parameter, the washers have inner and outer radii given by

$$x = R \pm \sqrt{r^2 - y^2}$$

The volume is therefore

$$V = \int_{-r}^{r} \pi \left[\left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2 \right] dy$$

Let us simplify the expression inside the brackets

$$\left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2$$

$$= R^2 + 2R\sqrt{r^2 - y^2} + (r^2 - y^2) - R^2 + 2R\sqrt{r^2 - y^2} - (r^2 - y^2) = 4R\sqrt{r^2 - y^2}$$

Thus

$$V = \int_{-r}^{r} 4\pi R\sqrt{r^2 - y^2}\, dy = 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2}\, dy$$

The last integral is an expression for the area of the semicircle bounded by $x^2 + y^2 = r^2$ and the y-axis. Thus it equals $\frac{1}{2}\pi r^2$. So finally

$$V = 4\pi R\left(\frac{1}{2}\pi r^2\right) = 2\pi^2 R r^2$$

67. The velocity is given by $v(t) = t^2 e^{-t}$. To find the position, we need to integrate this, and we use integration by parts.

$$\int t^2 e^{-t}\, dt = t^2(-e^{-t}) - \int 2t(-e^{-t})\, dt = -t^2 e^{-t} + 2\int te^{-t}\, dt$$

Using parts again

$$\int te^{-t}\, dt = t(-e^{-t}) - \int (-e^{-t})\, dt = -te^{-t} - e^{-t}$$

Putting it together

$$\int v(t)\, dt = -t^2 e^{-t} + 2[-te^{-t} - e^{-t}] + C = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + C$$

The problem asks how for the particle travels in the first t seconds. This is the definite integral

$$\int_0^t v(t')\, dt' = \left[-(t')^2 e^{-t'} - 2t'e^{-t'} - 2e^{-t'}\right]_0^t = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2$$

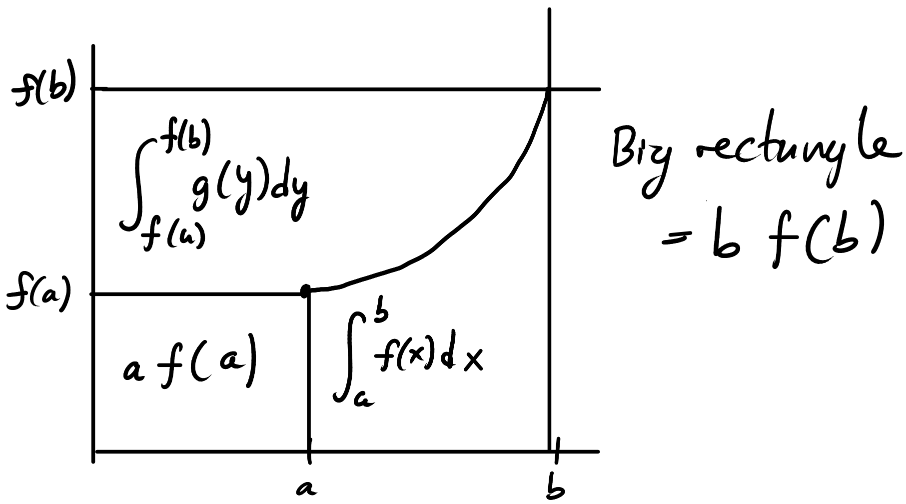
70. (a) We use integration by parts with $u = f(x)$ and $dv = 1\, dx$. Thus $du = f'(x)\, dx$ and $v = x$.

$$\int f(x)\, dx = xf(x) - \int xf'(x)\, dx$$

(b) Now we consider inverse functions f and g. We make the substitution $y = f(x)$ in the second integral. Thus $x = g(y)$ and $dy = f'(x)\,dx$. So $\int xf'(x)\,dx = \int g(y)\,dy$. If the limits of integration are $x = a$ to $x = b$, we must integrate from $y = f(a)$ to $y = f(b)$. Combining this with the previous part, we obtain

$$\int_a^b f(x)\,dx = [xf(x)]_a^b - \int_a^b xf'(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

(c) Here is a figure illustrating the identity in terms of areas:



(d) We evaluate $\int_1^e \ln x\,dx$. Here $f(x) = \ln x$ and $g(y) = e^y$. $f(1) = 0$ and $f(e) = 1$.

$$\int_1^e \ln x\,dx = e \cdot 1 - 1 \cdot 0 - \int_0^1 e^y\,dy = e - [e^y]_0^1 = e - (e-1) = 1$$

Lecture 23: Sections 7.1 and 7.2

48. (a) To prove the reduction formula, start with $\int \cos^n x\,dx$, and integrate by parts with $u = \cos^{n-1} x$ and $dv = \cos x\,dx$. Then $du = -(n-1)\cos^{n-2} x \sin x\,dx$, and $v = \sin x$. We obtain

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin x \sin x\,dx$$

Using $\sin^2 x = 1 - \cos^2 x$ we get

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x)\,dx$$

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

Putting the last term over on the left-hand side gives

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$$

And finally

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(b) We apply the formula for $n = 2$, using the fact that $\cos^0 x = 1$.

$$\int \cos^2 x\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

(c) We apply the formula for $n = 4$.

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx$$

Using the result of part (b),

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left[\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right] + C$$

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$$

67. To prove this formula, we can observe that $\sin mx \cos nx$ is an odd function of x. Therefore its integral over the symmetric interval $-\pi$ to π must be zero. More computationally, we can use the product-to-sum formula on page 476.

$$\sin mx \cos nx = \frac{1}{2}[\sin(mx - nx) + \sin(mx + nx)]$$

Thus

$$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\sin[(m-n)x] + \sin[(m+n)x]\}\,dx$$

As long as neither $m - n$ nor $m + n$ equals zero, we get

$$\frac{1}{2}\left[\frac{-\cos[(m-n)x]}{m-n} + \frac{-\cos[(m+n)x]}{m+n}\right]_{-\pi}^{\pi}$$

This equals zero because $\cos[(m-n)x]$ and $\cos[(m+n)x]$ are even functions, and so have the same values at $-\pi$ and π. If $m - n$ equals zero, then the first term is simply not present, as $\sin[0x] = 0$. Similarly, if $m + n$ equals zero, the the second term is not present. In all cases, the integral is zero.

68. The calculation is very similar to the previous problem.

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \{\cos[(m-n)x] - \cos[(m+n)x]\} \, dx$$

If neither $m - n$ nor $m + n$ equals zero, we get

$$\frac{1}{2} \left[\frac{\sin[(m-n)x]}{m-n} - \frac{\sin[(m+n)x]}{m+n} \right]_{-\pi}^{\pi}$$

This will always be zero because the value of sine at any integer multiple of π is zero.

In the statement of the problem we are to assume that m and n are positive integers, so we can ignore the possibility that $m + n$ could be zero (as then either m or n would be negative).

It remains to consider what happens if $m - n$ is zero, that is, if $m = n$. Then the term $\cos[(m-n)x] = \cos[0x] = 1$ in the integral reduces to a nonzero constant. We integrate this to

$$\frac{1}{2} \left[x - \frac{\sin[(m+n)x]}{m+n} \right]_{-\pi}^{\pi} = \frac{1}{2}[\pi - (-\pi)] = \pi$$

We conclude that for positive integers m and n, the integral is zero unless $m = n$, in which case it is π.

69. This is entirely analogous to the preceding problem. The only difference is that there is a plus sign between the two terms after we apply the product-to-sum formula.

Lecture 24: Section 7.3

37. We consider the region bounded by $y = 9/(x^2+9)$, $y = 0$, $x = 0$ and $x = 3$. We want to find the volume of the solid of rotation about the x-axis. We slice the solid vertically into disks, each with area $\pi(9/(x^2+9))^2$ and thickness dx. Thus

$$V = \int_0^3 \pi \frac{9^2}{(x^2+9)^2} \, dx = 81\pi \int_0^3 \frac{1}{(x^2+9)^2} \, dx$$

We use the substitution $x = 3\tan\theta$. Then $x^2 + 9 = 9\tan^2\theta + 9 = 9\sec^2\theta$, and $dx = 3\sec^2\theta \, d\theta$. We also convert the limits of integration: as θ goes from 0 to $\pi/4$, $\tan\theta$ covers exactly the interval from 0 to 1, and $x = 3\theta$ covers the interval from 0 to 3. Thus we convert the integral to

$$V = 81\pi \int_0^{\pi/4} \frac{3\sec^2\theta \, d\theta}{(9\sec^2\theta)^2} = 81\pi \int_0^{\pi/4} \frac{3}{81} \cos^2\theta \, d\theta = 3\pi \int_0^{\pi/4} \cos^2\theta \, d\theta$$

Now we use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, so we get

$$V = 3\pi\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} = 3\pi\left[\frac{\pi}{8} + \frac{1}{4}\sin(\pi/2)\right] = 3\pi\left[\frac{\pi}{8} + \frac{1}{4}\right] = \frac{3\pi^2}{8} + \frac{3\pi}{4}$$

39. (a) We apply the substitution $t = a\sin\theta$, $dt = a\cos\theta\, d\theta$:

$$\int \sqrt{a^2 - t^2}\, dt = \int \sqrt{a^2 - a^2\sin^2\theta}\, a\cos\theta\, d\theta = \int a^2\cos^2\theta\, d\theta$$

The original limits of integration are $t = 0$ to $t = x$. This becomes $\theta = 0$ to $\theta = \sin^{-1}(x/a)$.

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \int_0^{\sin^{-1}(x/a)} a^2\cos^2\theta\, d\theta = a^2\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\sin^{-1}(x/a)}$$

So we obtain

$$\frac{1}{2}a^2\sin^{-1}(x/a) + \frac{1}{4}a^2\sin(2\sin^{-1}(x/a))$$

We can write

$$\sin(2\sin^{-1}(x/a)) = 2\sin(\sin^{-1}(x/a))\cos(\sin^{-1}(x/a)) = 2(x/a)\sqrt{1-(x/a)^2}$$

Now we can put this into the prevous result and simplify to get

$$\frac{1}{4}a^2\sin(2\sin^{-1}(x/a)) = \frac{1}{4}a^2 2(x/a)\sqrt{1-(x/a)^2} = \frac{1}{2}x\sqrt{a^2 - x^2}$$

Putting it all together, we have

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \frac{1}{2}a^2\sin^{-1}(x/a) + \frac{1}{2}x\sqrt{a^2 - x^2}$$

(b) The figure shows the portion of the circle of radius a sitting over the interval from 0 to x. The integral $\int_0^x \sqrt{a^2 - t^2}\, dt$ equals A, the area underneath this curve. The figure shows that this area divided into two parts, a circular sector of angle θ, and a triangle whose vertices are $(0,0)$, $(x,0)$ and $(x, \sqrt{a^2 - x^2})$. The area of a circular sector is one-half radius-squared times angle, while the area of a triangle is one-half base times height. Thus

$$A = \frac{1}{2}a^2\theta + \frac{1}{2}x\sqrt{a^2 - x^2}$$

The figure also shows that $\sin\theta = x/a$, so $\theta = \sin^{-1}(x/a)$. Thus we have shown geometrically that

$$\int_0^x \sqrt{a^2 - t^2}\, dt = A = \frac{1}{2}a^2\sin^{-1}(x/a) + \frac{1}{2}x\sqrt{a^2 - x^2}$$

40. The parabola $y = \frac{1}{2}x^2$ divides the disk $x^2 + y^2 \le 8$ into two parts. Let us first compute the area of the smaller part which lies above the parabola. The two curves intersect when $y = \frac{1}{2}x^2$ and $x^2 + y^2 = 8$, thus $2y + y^2 = 8$. The solutions of the equation $y^2 + 2y - 8 = 0$ are

$$y = \frac{-2 \pm \sqrt{2^2 - 4(-8)}}{2} = \frac{-2 \pm \sqrt{36}}{2} = -1 \pm 3$$

As $y = \frac{1}{2}x^2$ must be positive, the negative solution is spurious, and we have $y = 2$. Thus $x^2 = 4$, and $x = \pm 2$. The area above the parabola and below the circle is given by the integral

$$A = \int_{-2}^{2} \left(\sqrt{8 - x^2} - \frac{1}{2}x^2 \right) dx = \int_{-2}^{2} \sqrt{8 - x^2}\, dx - \int_{-2}^{2} \frac{1}{2}x^2\, dx$$

To calculate the first integral, we can use the fact that the integrand is an even function, and the result of the previous problem with $a = \sqrt{8} = 2\sqrt{2}$.

$$\int_{-2}^{2} \sqrt{8 - x^2}\, dx = 2\int_{0}^{2} \sqrt{8 - x^2}\, dx = 2\left[\frac{1}{2} 8 \sin^{-1}(2/(2\sqrt{2})) + \frac{1}{2} 2\sqrt{8 - 2^2}\right]$$

$$= 8\sin^{-1}(1/\sqrt{2}) + 2\sqrt{2} = 8(\pi/4) + 2\sqrt{2} = 2\pi + 2\sqrt{2}$$

The second integral is straight-forward.

$$\int_{-2}^{2} \frac{1}{2}x^2\, dx = 2\int_{0}^{2} \frac{1}{2}x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

The total are of this part is

$$A = 2\pi + 2\sqrt{2} - \frac{8}{3}$$

The other part of the disk that is below the parabola has complementary area. Since the total area of the disk is 8π, this part must have area

$$8\pi - (2\pi + 2\sqrt{2} - 8/3) = 6\pi - 2\sqrt{2} + \frac{8}{3}$$

Lecture 25: Section 7.4

59. (a) We set $t = \tan(x/2)$ for $-\pi < x < \pi$. We draw a right triangle with angle $x/2$, adjacent leg 1, opposite leg t, and hypotenuse $\sqrt{1 + t^2}$. Then we have

$$\cos(x/2) = \frac{1}{\sqrt{1 + t^2}}, \quad \sin(x/2) = \frac{t}{\sqrt{1 + t^2}}$$

These expressions have the right sign for $-\pi < x < \pi$, since we have $\cos(x/2)$ positive in this range.

(b) We use the double angle formulas.

$$\cos x = \cos^2(x/2) - \sin^2(x/2) = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

$$\sin x = 2\sin(x/2)\cos(x/2) = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

(c) Solving $t = \tan(x/2)$ for x gives $x = 2\tan^{-1} t$. Thus

$$dx = \frac{2}{1+t^2}\,dt$$

60. We apply the substitution from the previous problem

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \int \frac{2}{(1+t^2)-(1-t^2)}\,dt = \int \frac{2}{2t^2}\,dt$$

$$= \int t^{-2}\,dt = -t^{-1} + C = -(\tan(x/2))^{-1} + C = -\cot(x/2) + C$$