SOLUTIONS TO UNGRADED HOMEWORK: LECTURES 1-8

M 408C, UNIVERSITY OF TEXAS AT AUSTIN

Lecture 1: Section 1.5

1. (a) 4 (b) $x^{-4/3}$
2. (a) 16 (b) $27x^7$
3. (a) $16b^{12}$ (b) $648y^7$
4. (a) x^{4n-3} (b) $a^{1/6}b^{-1/12}$

11-16. You may check work with a graphing calculator.

19. (a) Domain $= \{x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ (b) Domain = all real numbers $= (-\infty, \infty)$
20. (a) Domain = all real numbers $= (-\infty, \infty)$ (b) Domain = non-positive numbers $= (\infty, 0]$

Lecture 2: Section 1.6 and more

21. $f^{-1}(y) = [(y-1)^2 - 2]/3$
22. $f^{-1}(y) = (3y+1)/(-2y+4)$
23. $f^{-1}(y) = (\ln y + 1)/2$
24. $f^{-1}(y) = (1 + \sqrt{1+4y})/2$ Method: quadratic formula.
25. $f^{-1}(y) = e^y - 3$
26. $f^{-1}(y) = \ln(y/(1-2y))$ Method: first solve for e^x.
39. $\ln(5 \cdot 3^5) = \ln 1215$
40. $\ln[(a+b)(a-b)c^{-2}] = \ln \frac{a^2-b^2}{c^2}$
41. $\ln[(x+2)x^{1/2}(x^2+3x+2)^{-1}] = \ln \frac{(x+2)\sqrt{x}}{x^2+3x+2}$
75. The domain is where $3x+1$ is in the domain of $\sin^{-1}$, which is $[-1,1]$. Now $-1 \leq 3x+1 \leq 1 \iff -2 \leq 3x \leq 0 \iff -2/3 \leq x \leq 0$. So the domain is $[-2/3, 0]$. The range is the same as the range of $\sin^{-1}$, which is $[-\pi/2, \pi/2]$.

Part 2. Consider $f(x) = 1/(1+x)$. For any value of c, the equation $f(cx) = f(x)$ has a solution for x, namely $x = 0$, since obviously $f(c \cdot 0) = f(0)$. So the answer to the first question is "every value of c". For the second question, we are assuming that $f(cx) = f(x)$ has a solution for two values of x, which means that it has at least one solution other than zero. So we assume $x \neq 0$ solves this equation. Then we find $1/(1+cx) = 1/(1+x) \iff (1+x) = 1+cx \iff x = cx$. Since $x \neq 0$, we conclude that c must equal 1. So the answer to the second question is "only $c = 1$".

Part 3. If the function $g(x)$ satisfies $g(g(x)) = g(x)$, then $g(g(g(x)) = g(g(x)) = g(x)$ by applying the property twice. In general, if we denote by $g^{\circ n}$ the n-fold composition of g with itself, we find that $g^{\circ n} = g$ for any $n \geq 1$. We prove this by induction on n as follows. Take $n = 2$ as the base case: this says $g^{\circ 2} = g \circ g = g$, which was given. Now suppose that $g^{\circ(n-1)} = g$. We have $g^{\circ n} = g \circ g^{\circ(n-1)}$. We can replace $g^{\circ(n-1)}$ with g by the induction hypothesis, so we have $g^{\circ n} = g \circ g = g$. This completes the induction.

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Two obvious functions with this property are $g(x) = 0$, and $g(x) = x$. The latter is not constant. Let $g(x) = [x]$ denote the greatest integer function, which takes x to the greatest integer less than or equal to x. Then $g(g(x)) = x$, and the range of $g(x) = [x]$ is the set of integers.

It is impossible to find a function $g(x)$, such that $g(g(x)) = g(x)$, with $g(0) = 1$ and $g(1) = 0$. If we use $g(0) = 1$ and $g(1) = 0$, we find $g(g(0)) = g(1) = 0$, but if we use $g(g(x)) = g(x)$ and plug in $x = 0$, we find $g(g(0)) = g(0) = 1$.

1. Lecture 3: Section 2.2

6. (a) 4 (b) 4 (c) 4 (d) undefined (graph has an open circle there) (e) 1 (f) -1 (g) does not exist, because left and right limits do not agree (h) 1 (i) 2 (j) undefined (graph has an open circle there) (k) 3 (l) does not exist, because the function oscillates wildly.
11. For $x < -1$, the graph is a line of slope 1, passing through the point $(-1, 0)$. The graph for $-1 \leq x < 1$ is a parabola, going between the points $(-1, 1)$ and $(1, 1)$. Because these graphs do not match at $x = -1$, the limit does not exist there. The graph for $x \geq 1$ is a line of slope -1, passing through the point $(1, 1)$. Since this matches with the parabola, the limit exists at $x = 1$. So the limit exists everywhere except $x = -1$.
15. Consider
$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x = 0 \\ 2 & x > 0 \end{cases}$$
18. Consider
$$f(x) = \begin{cases} 2 & x \leq 0 \\ 3x/4 & 0 < x < 4 \\ 1 & x = 4 \\ 0 & 4 < x \end{cases}$$

2. Lecture 4: Sections 2.3 and 2.4

10. The equation
$$\frac{x^2 + x - 6}{x - 2} = x + 3$$
is not valid for every value of x. It is only valid if $x \neq 2$. The statement
$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} x + 3$$
is nevertheless valid, because the definition of $\lim_{x \to 2}$ does not depend on the value of either function at $x = 2$.
37. Suppose that $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for all $x \geq 0$. We want to find $\lim_{x \to 4} f(x)$. Then we know that, if $\lim_{x \to 4} f(x)$ exists,
$$\lim_{x \to 4} (4x - 9) \leq \lim_{x \to 4} f(x) \leq \lim_{x \to 4} (x^2 - 4x + 7)$$
Since we can find the limit of a polynomial by plugging in, we find
$$7 \leq \lim_{x \to 4} f(x) \leq 7$$
Thus, the squeeze theorem implies that this limit does indeed exist: $\lim_{x \to 4} f(x) = 7$.

46. Consider

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$$

This limit only depends on the behavior of the function for $x > 0$. For $x > 0$, we have $|x| = x$. Therefore, for $x > 0$, $1/x - 1/|x| = 0$. So this limit is the same as $\lim_{x\to 0^+}(0)$, which is zero.

60. We want to find functions $f(x)$ and $g(x)$ and a number a so that neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, but $\lim_{x\to a}[f(x)+g(x)]$ does exist. Just take $g(x) = -f(x)$, where $f(x)$ is some function whose limit does not exist. For example, $\lim_{x\to 0} 1/x$ does not exist, and neither does $\lim_{x\to 0} -1/x$, but $\lim_{x\to 0}(1/x-1/x) = \lim_{x\to 0} 0 = 0$ does exist.

1. Take $\delta = 0.1$. Then if $|x-1| < 0.1$, we have $0.9 < x < 1.1$. The given graph shows that this implies that $f(x)$ will be between 0.8 and 1.2, so $|f(x)-1| < 0.2$ as desired.

Lecture 5: Section 2.5

21. At $a = 0$, the limit from the left is $\cos 0 = 1$, while the limit from the right is $1 - 0^2 = 1$. Thus $\lim_{x\to 0} f(x) = 1$. However, $f(0) = 0$ by definition. The function is not continuous because its value at 0 differs from its limit at 0.

24. We may simplify

$$f(x) = \frac{x^3-8}{x^2-4} = \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \frac{x^2+2x+4}{x+2}$$

and this equation is valid if $x \neq 2$. Thus the limit of $f(x)$ is

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3$$

Thus if we define $f(2) = 3$, we will remove the discontinuity at $x = 2$. (Note, although it is not part of the problem, that there is still a discontinuity at $x = -2$.)

30. The function $\tan x$ is continous on its domain, and its domain is all numbers except the numbers of the form $\frac{\pi}{2} + n\pi$ for an integer n. The function $4 - x^2$ is continous everywhere because it is a polynomial, and the function $\sqrt{u}$ is continous on its domain. Thus the function $\sqrt{4-x^2}$ is continous on its domain, because it is the composition of these continous functions. The domain of $\sqrt{4-x^2}$ is $[-2, 2]$. The quotient of continous functions is continous whereever the denominator is not zero. Thus $\tan x/\sqrt{4-x^2}$ is continous on its domain. The domain consists of the numbers in the interval $(-2, 2)$ not of the form $\frac{\pi}{2}+n\pi$. We must throw away -2 and 2 because the denominator is zero there. In fact, the only discontinuities of $\tan x$ that lie in the interval $(-2, 2)$ are $\pi/2$ and $-\pi/2$, so we find that the domain is

$$(-2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 2)$$

and the function is continuous on this domain.

38. We want to find

$$\lim_{x\to 2}\arctan\left(\frac{x^2-4}{3x^2-6x}\right)$$

Note that the rational function inside the arctan can be simplified by canceling a factor of $x - 2$, assuming $x \neq 2$:

$$\frac{x^2-4}{3x^2-6x} = \frac{x+2}{3x}$$

Because the definition of the limit doesn't depend on what happens at $x = 2$, we get

$$\lim_{x\to 2} \arctan\left(\frac{x^2-4}{3x^2-6x}\right) = \lim_{x\to 2} \arctan\left(\frac{x+2}{3x}\right)$$

In the right-hand side, the function is continuous at 2, because $(x+2)/3x$ is continuous at 2, and arctan is continuous everywhere. Thus we can plug in to compute the limit.

$$\lim_{x\to 2} \arctan\left(\frac{x+2}{3x}\right) = \arctan(4/6) = \arctan(2/3)$$

51. Consider the function $f(x) = x^4 + x - 3$. This function is continuous because it is a polynomial. The values at 1 and 2 are $f(1) = -1$, and $f(2) = 15$. Since $f(1) < 0 < f(2)$, there must be some value c in the interval $(1, 2)$ such that $f(c) = 0$, by the intermediate value theorem. Thus the equation $x^4 + x - 3 = 0$ has a solution in the interval $(1, 2)$.

3. Lecture 6: Section 2.7 and 2.8

11. (a) Moving to the right if $0 < t < 1$ or $4 < t < 6$. Moving to the left if $2 < t < 3$. Standing still if $1 < t < 2$ or $3 < t < 4$. (b) the velocity function is

$$v(t) = \begin{cases} 3 & 0 < t < 1 \\ 0 & 1 < t < 2 \\ -2 & 2 < t < 3 \\ 0 & 3 < t < 4 \\ 1 & 4 < t < 6 \end{cases}$$

17. $g'(0)$ is least because it is negative. 0 is the next in the sequence, because all the other numbers are positive. Judging by the graph, $g'(4)$ is next because the slope is not very steep there. Then $g'(2)$ is next, and $g'(-2)$ is the greatest, because the graph is pretty steep there.
19. The equation of the tangent line to $y = f(x)$ at $a = 2$ has the form $y - f(2) = f'(2)(x-2)$, or
$$y = f'(2)x - 2f'(2) + f(2)$$
We are told the equation of the tangent line is $y = 4x - 5$. Thus we find $f'(2) = 4$, and $-2f'(2) + f(2) = -5$. We can solve the equation for $f(2)$, to get $f(2) = -5 + 2f'(2) = -5 + 2 \cdot 4 = 3$
3. (a) II: The slope starts out very negative, then increases until $x = 0$, where it starts decreasing symmetrically. (b) IV: the function is piecewise linear, so its derivative should be piecewise constant. (c) I: On the left side of the graph, the slope is small negative, in the middle it is zero, and on the right side of the graph it is small positive. (d) III: On left, slope is big positive, on right it is big negative.
11. For $x < 0$, the derivative increases until we get a vertical tangent (infinite slope) at $x = 0$, so $f'(x)$ approaches $+\infty$ from the left. For $x > 0$, the slope is always positive, and decreasing. The limit of $f'(x)$ from the right at 0 is also $+\infty$.
12. The graph of $P'(t)$ starts close to zero, increases for a while, then hits a maximum and starts decreasing and approaching zero. This means that the yeast population is always increasing; at first the growth accelerates, but eventually it starts to slow down.
43. f is a, f' is b, and f'' is c. We find that b is positive precisely when a is increasing, so this shows that b is the derivative of a. Also, we find that c is positive precisely when b is increasing, so this shows that c is the derivative of b.

4. Lecture 7: Sections 3.1 and 3.2

51. Where is the tangent to $y = 2x^3 + 3x^2 - 12x + 1$ horizontal? The derivative is $y' = 6x^2 + 6x - 12$. The tangent line is horizontal if this is zero, so we solve

$$y' = 6x^2 + 6x - 12 = 0$$
$$x^2 + x - 2 = 0$$
$$x = \frac{-1 \pm \sqrt{1 - 4(1)(-2)}}{2} = \frac{-1 \pm \sqrt{9}}{2} = 1, -2$$

The tangent line is horizontal at the points $(1, -6)$ and $(-2, 21)$.

53. The curve $y = 2e^x + 3x + 5x^3$ has a tangent line of slope 2 if and only if $2 = y' = 2e^x + 3 + 15x^2$. Rearranging the terms, this is the same as saying

$$2e^x = -1 - 15x^2$$

This equation cannot have any solution, because $-1 - 15x^2$ is negative for every value of x, while $2e^x$ is positive for every value of x. So the curve has no tangent line of slope 2.

54. Find the equation of the tangent line to $y = x\sqrt{x}$ parallel to $y = 1 + 3x$. This just means to find the tangent line with slope 3. Note that $y = x^{3/2}$, so $y' = (3/2)x^{1/2}$. We solve

$$3 = y' = (3/2)x^{1/2}$$
$$2 = x^{1/2}$$
$$4 = x$$

This means that the tangent line of slope 3 occurs at the point $x = 4, y = 8$. The equation for this line is then $(y - 8) = 3(x - 4)$.

54. Find equations of tangent lines to $y = \frac{x-1}{x+1}$ that are parallel to $x - 2y = 2$. First of all, the given line also has equation $y = (1/2)x - 1$, so it has slope $1/2$. The derivative of y is given by the quotient rule.

$$y' = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

This equals $1/2$ when $(x+1)^2 = 4$. Thus we find $x + 1 = \pm 2$, and $x = -3$ or 1. At $x = -3$, $y = 2$, so the tangent line is $(y - 2) = (1/2)(x + 3)$. At $x = 1$, $y = 0$, so the tangent line is $y = (1/2)(x - 1)$.

55. Consider $R(x)$

$$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$$

We want to find $R'(0)$. We know by the quotient rule that

$$R'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{g(0)^2}$$

Where $f(x) = x - 3x^3 + 5x^5$ and $g(x) = 1 + 3x^3 + 6x^6 + 9x^9$. Thus $f(0) = 0$, and $g(0) = 1$.

$$f'(x) = 1 - 9x^2 + 25x^4$$

so $f'(0) = 1$, and

$$g'(x) = 9x^2 + 36x^5 + 81x^8$$

so $g'(0) = 0$. Thus

$$R'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{g(0)^2} = \frac{1 \cdot 1 - 0 \cdot 0}{1^2} = 1$$

59. (a) Consider $(fgh)'$. Using the product rule

$$(fgh)' = f'(gh) + f(gh)' = f'gh + f(g'h + gh') = f'gh + fg'h + fgh'$$

(b) taking $f = g = h$, we find

$$(f^3)' = f'f^2 + ff'f + f^2f' = 3f^2f'$$

which in a different notation is

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) To differentiate $y = e^{3x}$, we note that $e^{3x} = (e^x)^3$, so we use $f(x) = e^x$. Then $f'(x) = e^x$ as well, and

$$y' = 3f(x)^2 f'(x) = 3(e^x)^2 e^x = 3e^{3x}$$

5. Lecture 8: Section 3.3

31. (a) We want to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

using the quotient rule. The derivative of the top is $\frac{d}{dx}(\tan x - 1) = \sec^2 x$, while the derivative of the bottom is $\frac{d}{dx}(\sec x) = \sec x \tan x$. Thus we find

$$f'(x) = \frac{(\sec x)(\sec^2 x) - (\tan x - 1)(\sec x \tan x)}{\sec^2 x}$$

(b) Now we differentiate by writing everything in terms of $\sin x$ and $\cos x$. We find

$$f(x) = \frac{\tan x - 1}{\sec x} = \frac{\frac{\sin x}{\cos x} - 1}{\frac{1}{\cos x}} = \left(\frac{\sin x}{\cos x} - 1\right)\cos x = \sin x - \cos x$$

Thus $f'(x) = \cos x + \sin x$. (c) We want to show that the two answers are equivalent. Starting with the first one, we can cancel a factor of $\sec x$ from all of the terms.

$$f'(x) = \frac{\sec^2 x - (\tan x - 1)\tan x}{\sec x}$$

expand the top

$$\frac{\sec^2 x - \tan^2 x + \tan x}{\sec x}$$

Use the Pythagorean theorem in the form $1 + \tan^2 x = \sec^2 x$ to get

$$\frac{1 + \tan x}{\sec x}$$

write everything in terms of $\sin x$ and $\cos x$.

$$\frac{1 - \frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \left(1 - \frac{\sin x}{\cos x}\right)\cos x = \cos x + \sin x$$

and this was the other expression for $f'(x)$.

49. To look for the pattern, we compute

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d^2}{dx^2}(\sin x) = \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d^3}{dx^3}(\sin x) = \frac{d}{dx}(-\sin x) = -\cos x$$
$$\frac{d^4}{dx^4}(\sin x) = \frac{d}{dx}(-\cos x) = \sin x$$

This means that if we take the derivative of $\sin x$ four times, the result is $\sin x$ itself. If we take the derivative four more times, we will get $\sin x$ again, so $\frac{d^8}{dx^8}(\sin x) = \sin x$. Similarly, the twelfth, sixteenth, twentieth, etc. derivatives of $\sin x$ are $\sin x$ itself. Now we write $99 = 4 \cdot 24 + 3$. When we take the derivative of $\sin x$ $96 = 4 \cdot 24$ times, we get $\sin x$ back, and then we just have to take the derivative three more times. So

$$\frac{d^{99}}{dx^{99}}(\sin x) = -\cos x$$