Review

FINAL: MONDAY, DEC. 16, 2:00–5:00 pm

in CPE 2.214. (usual room)

Hui Yu's Review Office Hours:

WED + THUR 3:30 – 5:00 pm

JP's Office Hours Thur + Fri 10:30–12:00 noon

Final Format: $\frac{3}{4}$ Multiple Choice, $\frac{1}{4}$ Free Response.

Cumulative, but with some emphasis.

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Topics:
- Limits, Limit rules, L'Hopital's rule, squeeze theorem.
  - Inverse functions + $\ln$ + $\sin^{-1}$ + $\tan^{-1}$
- Continuity, Intermediate value theorem.
- Derivatives, slope of tangent line.

  Definition $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

  velocity + acceleration.
- Differentiation Rules, Derivatives of trig functions
- Chain rule, implicit differentiation

- Related rates.
- Linear approximation, differentials
- Optimization (Max/Min)
- Extreme value theorem.
- Fermat's theorem at a max/min $x=c$

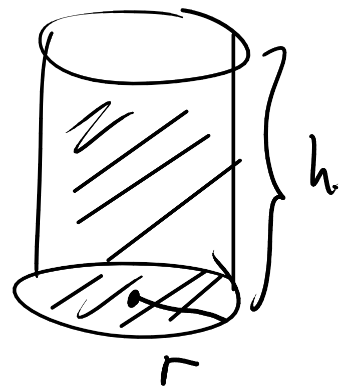
critical point: $f'(c) = 0$ or DNE.

Absolute max/min: critical points,
check endpoints or limits at $\pm\infty$

- Rolle's theorem + Mean Value theorem
- 1st + 2nd derivative tests for max/min
  - → concavity / inflection pts
- <u>Integration</u> Area, Riemann sums, FTC
  - ↳ Areas between curves, Volumes
- Integration techniques: substitution, int. by parts
- Trig integrals, trig substitution, partial fraction

Optimization:

Problem: cylindrical can with out a top




made to contain $V$ $cm^3$ of liquid. What is minimal surface area.

constant $= V = \pi r^2 h$ $\qquad A = 2\pi r h + \pi r^2$

$$\frac{V}{\pi r^2} = h \longrightarrow A = 2\pi r\left(\frac{V}{\pi r^2}\right) + \pi r^2$$

$$A = \frac{2V}{r} + \pi r^2$$

Domain $r$ in $(0,\infty)$

$$\frac{dA}{dr} = 2V\frac{-1}{r^2} + 2\pi r = 0$$

$$-\frac{2V}{r^2} + 2\pi r = 0 \iff 2\pi r = \frac{2V}{r^2}$$

$$\iff \pi r^3 = V \iff r = \sqrt[3]{\frac{V}{\pi}}$$

$$r = \sqrt[3]{\frac{V}{\pi}} \qquad h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\frac{V}{\pi}\right)^{2/3}} = \frac{V}{\pi}\,\frac{\pi^{2/3}}{V^{2/3}}$$

$$= \frac{V^{1/3}}{\pi^{1/3}} = \sqrt[3]{\frac{V}{\pi}}$$

$\frac{dA}{dr}$ $\quad - \quad | \quad + \quad r$

$r = \sqrt[3]{\frac{V}{\pi}}$

$r > \sqrt[3]{\frac{V}{\pi}} \qquad \pi r^3 > V \qquad 2\pi r > \frac{2V}{r^2}$

$$\frac{dA}{dr} = 2\pi r - \frac{2V}{r^2} > 0$$

This is a local minimum.

$$\lim_{r \to 0} A = \lim_{r \to 0} \frac{2V}{r} + 2\pi r^2 = +\infty$$

$$\lim_{r \to \infty} A = \lim_{r \to \infty} \frac{2V}{r} + 2\pi r^2 = +\infty$$

$$\int \frac{6}{x\sqrt{x^2+4}}\, dx$$

$x = 2\tan u \qquad x^2+4 = (2\tan u)^2 + 4 = 4\tan^2 u + 4$

$dx = 2\sec^2 u\, du \qquad = 4(\tan^2 u + 1) = 4\sec^2 u$

$$\int \frac{6 \cdot 2\sec^2 u\, du}{2\tan u \; 2\sec u} = 3\int \frac{\sec u}{\tan u}\, du$$

$$= 3\int \frac{1}{\cos u}\,\frac{\cos u}{\sin u}\,du = 3\int \frac{1}{\sin u}\,du = 3\int \csc u\,du$$

$$\int \sec u\,du = \ln\left|\sec u + \tan u\right| + C$$

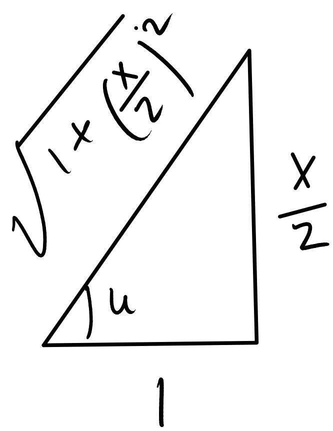
$$\int \csc u \cdot \frac{\csc u + \cot u}{\csc u + \cot u}\,du$$

$$v = \csc u + \cot u$$

$$dv = -\csc u\cot u - \csc^2 u = -\csc u(\cot u + \csc u)$$

$$= \int -\frac{1}{v}\,dv = -\ln|v| = -\ln\left|\csc u + \cot u\right| + C$$

$$u = \tan^{-1}\frac{x}{2}$$




$$\csc u = \frac{\sqrt{1+\left(\frac{x}{2}\right)^2}}{\left(\frac{x}{2}\right)}$$

$$\cot u = \frac{2}{x}$$

$$= -3\ln\left|\frac{\sqrt{1+\left(\frac{x}{2}\right)^2}}{\left(\frac{x}{2}\right)} + \frac{2}{x}\right|$$

$$\frac{\sqrt{1+\left(\frac{x}{2}\right)^2}}{\left(\frac{x}{2}\right)} = \frac{2\sqrt{1+\left(\frac{x}{2}\right)^2}}{x} = \frac{\sqrt{4+4\left(\frac{x}{2}\right)^2}}{x} = \frac{\sqrt{4+x^2}}{x}$$

$$-3\ln\left|\frac{\sqrt{4+x^2}}{x}+\frac{2}{x}\right| = -3\ln\left|\frac{\sqrt{4+x^2}+2}{x}\right|$$

$$= 3\ln\left|\frac{x}{\sqrt{4+x^2}+2}\right| = 3\ln\left|\frac{\sqrt{4+x^2}-2}{x}\right|$$

$$\frac{x}{\sqrt{4+x^2}+2}\cdot\frac{\sqrt{4+x^2}-2}{\sqrt{4+x^2}-2} = \frac{x\left(\sqrt{4+x^2}-2\right)}{4+x^2-4}$$

$$= \frac{\sqrt{4+x^2}-2}{x}$$

$$\int \frac{dx}{\sqrt{\sqrt{x}+1}} \qquad u=\sqrt{\sqrt{x}+1} \qquad u^2=\sqrt{x}+1$$

$$u^2-1=\sqrt{x} \qquad (u^2-1)^2=x$$

$$dx = 2(u^2-1)\,2u\,du$$

$$\int \frac{1}{u}\,2(u^2-1)\,2u\,du = 4\int(u^2-1)\,du$$

$$= 4\left[\frac{u^3}{3}-u\right] = \frac{4}{3}\left(\sqrt{\sqrt{x}+1}\right)^3 - 4\left(\sqrt{\sqrt{x}+1}\right)+C$$

$$\int \sin^{-1}\sqrt{x}\,dx \qquad u = \sin^{-1}\sqrt{x}$$

$$\sin u = \sqrt{x}$$

$$\sin^2 u = x$$

$$2\sin u\cos u\,du = dx$$

$$= \int u\, 2\sin u\cos u\,du = \int u\sin 2u\,du$$

$$f(u) = u \qquad g(u) = -\frac{1}{2}\cos 2u$$

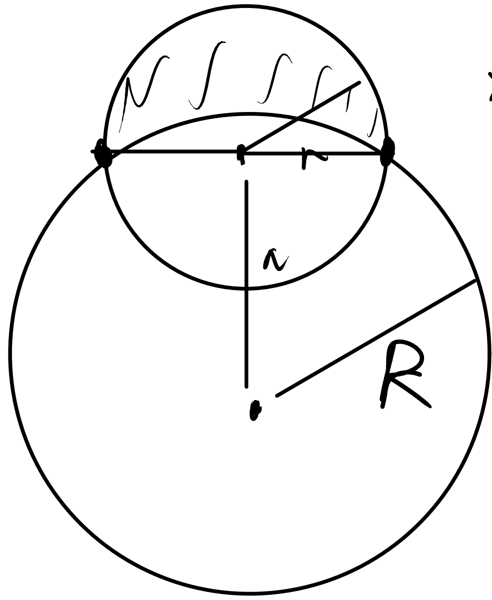
$$f'(u) = 1 \qquad g'(u) = \sin 2u$$

$$= -\frac{1}{2}u\cos 2u - \int -\frac{1}{2}\cos 2u\,du$$

$$= -\frac{1}{2}u\cos 2u + \frac{1}{2}\,\frac{1}{2}\sin 2u + C$$

$$= -\frac{1}{2}u\cos 2u + \frac{1}{4}\sin 2u + C$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}\,\cos\left(2\sin^{-1}\sqrt{x}\right) + \frac{1}{4}\sin\left(2\sin^{-1}\sqrt{x}\right) + C$$




$$x^2+(y-a)^2=r^2$$

$$y=a+\sqrt{r^2-x^2}$$

$$x^2+y^2=R^2$$

$$y=\sqrt{R^2-x^2}$$

$$\int_{-r}^{r}\left(a+\sqrt{r^2-x^2}-\sqrt{R^2-x^2}\right)dx$$

$$\int_{-r}^{r}\sqrt{R^2-x^2}\,dx$$

$$x=R\sin\theta$$
$$dx=R\cos\theta\,d\theta$$
$$\theta=\sin^{-1}\frac{x}{R}$$

$$\int\sqrt{R^2-(R\sin\theta)^2}\;R\cos\theta\,d\theta$$

$$=\int R\cos\theta\;R\cos\theta\,d\theta=R^2\int\cos^2\theta\,d\theta$$

$$=R^2\int\frac{1+\cos2\theta}{2}\,d\theta=R^2\left[\frac{\theta}{2}+\frac{\sin2\theta}{4}\right]_{\theta=\sin^{-1}\left(\frac{-r}{R}\right)}^{\theta=\sin^{-1}\left(\frac{r}{R}\right)}$$

$$\int\sin^3u\,du=\int\sin^2u\;\sin u\,du=\int(1-\cos^2u)\sin u\,du$$

$v = \cos u \quad dv = -\sin u \, du \quad = -\int (1-v^2)\, dv$

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx$$ complete square

$$\int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$x-1 = \sin u$

$dx = \cos u \, du$

$$\int \frac{1}{\sqrt{1-\sin^2 u}} \cos u \, du = \int \frac{\cos u}{\sqrt{\cos^2 u}}\, du = \int du = u + C$$

$$= \sin^{-1}(x-1) + C$$