$\boxed{\text{Partial Fractions}}$

Thursday - Review : Bring Questions

<u>Theorem</u>: every rational function can be integrated in elementary terms

Rational function : ratio of two polynomials

$$\frac{2x^7+8x^6+13x^5+20x^4+17x^3+16x^2+7x+7}{(x^2+x+1)^2(x^2+2x+2)(x-1)^2}$$

$$= \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x^2+x+1)^2} + \frac{x+3}{x^2+2x+2}$$

$\int \downarrow$ $\ln|x-1|$

$\int \downarrow$ $\frac{-2}{x-1}$

$\downarrow$ complete square trig substitution. or other tricks.

Long division of polynomials
use this if degree of numerator ≥ degree of denominator

$$\frac{x^3+3x^2+3x+2}{x+1}$$

$$\begin{array}{r|l} & x^2+2x+1 \\ \hline x+1 & x^3+3x^2+3x+2 \\ & x^3+x^2 \\ & \quad 2x^2+3x+2 \\ & \quad 2x^2+2x \\ & \qquad x+2 \\ & \qquad x+1 \end{array}$$

Quotient = x^2+2x+1
remainder = 1

$$\frac{x^3+3x^2+3x+2}{x+1} = x^2+2x+1+\frac{1}{x+1}$$

$$\int \frac{x^3+3x^2+3x+2}{x+1}\,dx = \int\left[x^2+2x+1+\frac{1}{x+1}\right]dx$$

$$= \frac{x^3}{3}+x^2+x+\ln|x+1|+C$$

Case I: Denominator is a product of linear factors, none of which are repeated

$$R(x) = \frac{P(x)}{Q(x)} \qquad Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_rx+b_r)$$

Then $R(x) = \dfrac{A_1}{a_1x+b_1} + \dfrac{A_2}{a_2x+b_2} + \cdots + \dfrac{A_r}{a_rx+b_r}$

Ex $\dfrac{10x^2+11x-4}{3x^3+5x^2-2x}$

$$Q(x) = 3x^3+5x^2-2x$$
$$= x(3x^2+5x-2)$$
$$= x(3x-1)(x+2)$$

$$\frac{10x^2+11x-4}{x(3x-1)(x+2)} = \frac{A}{x}+\frac{B}{3x-1}+\frac{C}{x+2}$$

How to determine A, B, C?

Clear denominator mult' by $x(3x-1)(x+2)$

$$10x^2+11x-4 = A(3x-1)(x+2)+Bx(x+2)+Cx(3x-1)$$

$$= A(3x^2+5x-2)+B(x^2+2x)+C(3x^2-x)$$

$$=(3A+B+3C)x^2+(5A+2B-C)x+(-2A)$$

$$10 = 3A+B+3C$$
$$11 = 5A+2B-C$$
$$-4 = -2A$$

$A=2$

$$10 = 6+B+3C \qquad 4 = B+3C \qquad B = 4-3C$$
$$11 = 10+2B-C \qquad 1 = 2B-C$$
$$1 = 2(4-3C)-C$$
$$1 = 8-6C-C$$
$$-7 = -7C \Rightarrow C = 1$$

$$B = 4-3(1) = 1$$

$$\frac{10x^2+11x-4}{x(3x-1)(x+2)} = \frac{2}{x}+\frac{1}{3x-1}+\frac{1}{x+2}$$

$$\int\left[\frac{2}{x}+\frac{1}{3x-1}+\frac{1}{x+2}\right]dx$$

$$2\ln|x|+\frac{1}{3}\ln|3x-1|+\ln|x+2|+C$$

Another way

$$10x^2+11x-4 = A(3x-1)(x+2)+Bx(x+2)+Cx(3x-1)$$

Plug in $x=0$

$$-4 = A(-1)(2)+B\,0(2)+C(0)(-1)$$

$$-4=-2A \qquad A=2$$

Plug in $x=-2$

$$10(-2)^2+11(-2)-4 = A\cdot 0+B\cdot 0+C(-2)(3(-2)-1)$$

$$40-22-4 = C\cdot 14$$

$$40-26 = 14C \qquad C=1$$

$$14 = 14C$$

Plug in $x=\frac{1}{3} \Rightarrow (3x-1)=0$

Try it get $B=1$

Case II repeated liner factors

If you see $\frac{1}{(ax+b)^r}$ in the denominator

you'll need $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_r}{(ax+b)^r}$

$R(x)$

Ex $R(x) = \dfrac{x^4-2x^2+4x+1}{x^3-x^2-x+1} \overset{\text{long div}}{=} x+1 + \dfrac{4x}{x^3-x^2-x+1}$

Focus on $\dfrac{4x}{x^3-x^2-x+1}$

x^3-x^2-x+1

$(1)^3-(1)^2-1+1=0$

divisible by $x-1$

$= (x-1)(x^2-1) = (x-1)(x-1)(x+1)$

$= (x-1)^2(x+1)$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Plug in 1 : $4 = 0 + 2B + C\cdot 0$

$B = 2$

Plug in -1 : $-4 = 0 + 0 + C(-1-1)^2$

$-4 = C\cdot 4$

$C = -1$

Plug in 0 :

$0 = A(-1)(1) + B(1) + C(-1)^2$

$0 = -A + B + C$

$0 = -A + 2 - 1$ $\quad A = 1$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

$$\int R(x)\,dx = \int \left[x+1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{(x-1)} - \ln|x+1| + C$$

<u>Fact</u> any polynomial $Q(x)$ factors essentially uniquely into linear factors and irreducible quadratic factors

<u>eg.</u> $x^2+1 \qquad ax^2+bx+c \;,\; b^2-4ac<0$

$\quad \parallel$

$(x+i)(x-i)$

$i=\sqrt{-1}$

Case III irreducible quadratic factor, w/o repetition

$\dfrac{1}{ax^2+bx+c}$ need $\dfrac{Ax+B}{ax^2+bx+C}$ in the partial fraction.

Eg. $\dfrac{2x^2-x+4}{x^3+4x} \qquad Q(x) = x^3+4x = x(x^2+4)$

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

Plug in 0: $\quad 4 = 4A + 0 \quad A = 1$

$$2x^2 - x + 4 = x^2 + 4 + (Bx+C)x$$

$$x^2 - x = (Bx+C)x$$

$$x - 1 = Bx + C \qquad B = 1 \quad C = -1$$

$$R(x) = \frac{2x^2 - x + 4}{x(x^2+4)} = \frac{1}{x} + \frac{x-1}{x^2+4}$$

$$= \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}$$

$$\int R(x)\,dx = \int \left[\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right] dx$$

$$\int \frac{1}{x}\,dx = \ln|x|$$

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+4|$$

$u = x^2+4$

$du = 2x\,dx$

$$\int \frac{1}{x^2+4}\,dx \qquad x = 2\tan\theta \qquad dx = 2\sec^2\theta\,d\theta$$

$$x^2+4 = (2\tan\theta)^2 + 4$$

$$= 4\tan^2\theta + 4$$

$$= 4(\tan^2\theta + 1)$$

$$= 4(\sec^2\theta)$$

$$= \int \frac{1}{4\sec^2\theta}\, 2\sec^2\theta\,d\theta = \frac{1}{2}\int d\theta$$

$= \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}\frac{x}{2}$

$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$\int \frac{1}{\sqrt{a^2-x^2}}dx \qquad x = a\sin x$

$Ans = \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

Case IV quadratic factor repeated.

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{1}{x} - \frac{x+1}{(x^2+1)} + \frac{x}{(x^2+1)^2}$$

$$\int \frac{x}{(x^2+1)^2}dx = \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\,\frac{-1}{u} = \frac{-1}{2(x^2+1)}$$

$u = x^2+1$

$du = 2x\,dx$

$$\int \frac{-(x+1)}{(x^2+1)} = -\int \frac{x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$$

$$= -\frac{1}{2}\ln|x^2+1| - \tan^{-1}x$$

$$\text{Ans} = \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$