HW due next Thursday

No Discussion session tomorrow.

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Trig substitution: Kind of a reverse sub.

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$$

try $u=\sqrt{e^x+1}$

$$du = \frac{1}{2\sqrt{e^x+1}} e^x\,dx$$

$u=\sqrt{e^x+1}$ solve for x

$u^2 = e^x+1$

$u^2-1 = e^x$

$\ln(u^2-1) = x$

$$dx = \frac{1}{u^2-1}\,2u\,du$$

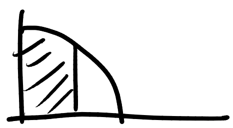
$$e^{2x} = (e^x)^2 = (u^2-1)^2$$

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \int \frac{(u^2-1)^2}{u}\,\frac{2u}{u^2-1}\,du$$

$$= \int 2(u^2-1)\,du = \frac{2}{3}u^3 - 2u + C$$

$$= \frac{2}{3}\left(\sqrt{e^x+1}\right)^3 - 2\sqrt{e^x+1} + C$$

$$\int \sqrt{1-x^2}\,dx \qquad \text{Try } x = \sin u$$



(same as $u = \sin^{-1} x$)

$$dx = \cos u \, du$$

$$\int \sqrt{1-\sin^2 u}\,\cos u \, du = \int \sqrt{\cos^2 u}\,\cos u \, du$$

$$= \int \cos u \cos u \, du = \int \cos^2 u \, du$$

$$\cos^2 u = \frac{1+\cos 2u}{2} \qquad \cos 2u = 2\cos^2 u - 1$$

$$= \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2} + \frac{\sin 2u}{4} + C$$

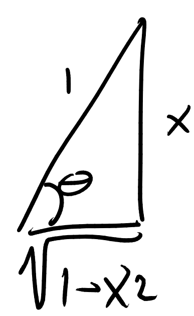
$$= \frac{\sin^{-1} x}{2} + \frac{\sin(2\sin^{-1} x)}{4} + C$$

$$\sin 2A = 2 \sin A \cos A$$

$$\sin(2\sin^{-1} x) = 2\sin(\sin^{-1} x)\cos(\sin^{-1} x)$$

$$= 2x\sqrt{1-x^2}$$

$$\text{Ans} = \frac{\sin^{-1} x}{2} + \frac{1}{2}x\sqrt{1-x^2} + C$$



Trig substitution is based on the pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad\qquad a^2 - (a\sin\theta)^2 = a^2 - a^2\sin^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad\qquad = a^2(1-\sin^2\theta)$$

$$= a^2\cos^2\theta$$

| If you see... | Try... | get... |
|---|---|---|
| $a^2 - x^2$ | $x = a\sin\theta$ | $a^2 - (a\sin\theta)^2 = a^2\cos^2\theta$ |
| $a^2 + x^2$ | $x = a\tan\theta$ | $a^2 + (a\tan\theta)^2 = a^2\sec^2\theta$ |
| $x^2 - a^2$ | $x = a\sec\theta$ | $(a\sec\theta)^2 - a^2 = a^2\tan^2\theta$ |

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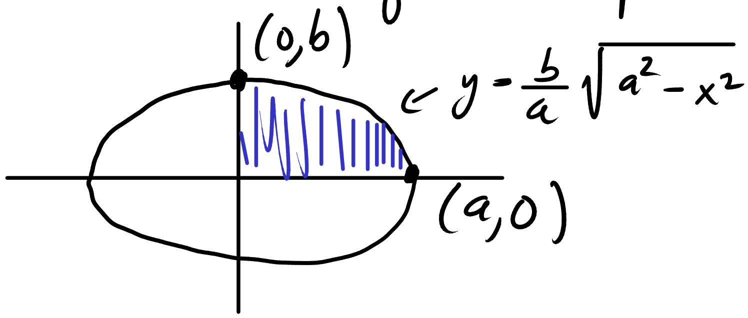
Find the area of an ellipse $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$




$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \qquad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \sqrt{\frac{b^2}{a^2}(a^2-x^2)} = \frac{b}{a}\sqrt{a^2-x^2}$$

$$\frac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$x = a\sin\theta$ $\qquad$ $a^2 - x^2 = a^2 - (a\sin\theta)^2$

$dx = a\cos\theta\,d\theta$ $\qquad$ $= a^2 - a^2\sin^2\theta$

$= a^2(1-\sin^2\theta) = a^2\cos^2\theta$

limits $\quad x = 0 \Longleftrightarrow a\sin\theta = 0 \Leftrightarrow \theta = 0$

$x = a \Longleftrightarrow a\sin\theta = a \Leftrightarrow \theta = \frac{\pi}{2}$

$$\frac{1}{4}A = \int_0^{\pi/2} \frac{b}{a}\sqrt{a^2\cos^2\theta}\; a\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{b}{a}\, a\cos\theta\, a\cos\theta\,d\theta = ab\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= ab\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = ab\left[\frac{\pi}{4} + \frac{0}{4} - \frac{0}{2} - \frac{0}{4}\right]$$

$$= \frac{\pi}{4}ab$$

$$A = \pi ab$$

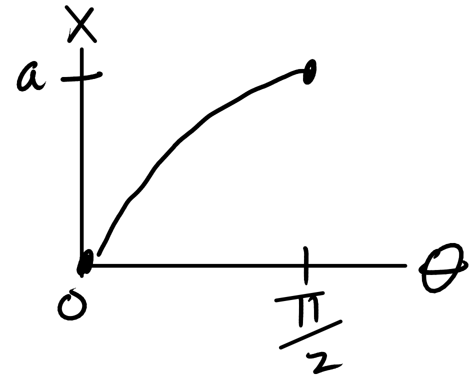
Why did I choose $\theta : 0 \to \frac{\pi}{2}$

Had $x$ goes from $0 \to a$ $\qquad$ $x = a\sin\theta$

$\theta = 0 \Rightarrow x = 0$

$\theta = \frac{\pi}{2} \Rightarrow x = a$

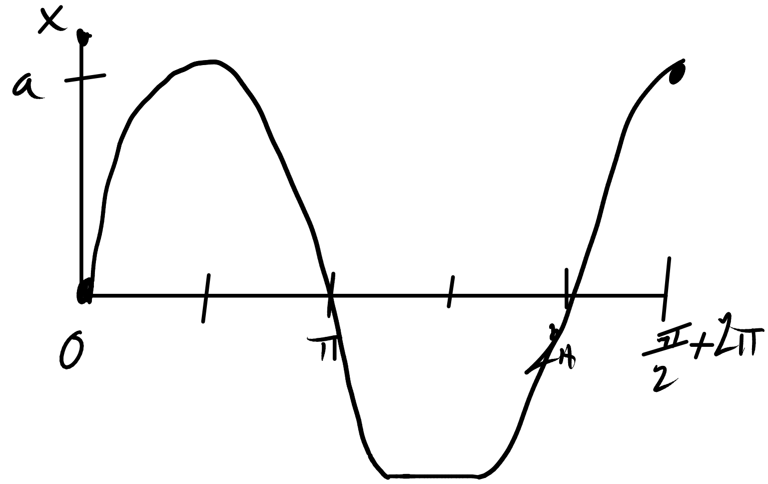
Plot $x$ vs $\theta$




why not $\theta = 0$ to $\theta = \frac{\pi}{2} + 2\pi$

$$\sin\left(\frac{\pi}{2} + 2\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

plot $x$ vs $\theta$



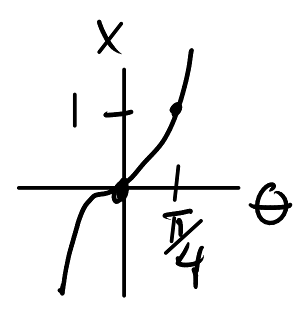

$$\int_0^1 \frac{dx}{(x^2+1)^2}$$

$x = \tan\theta$ $\quad x^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$dx = \sec^2\theta \, d\theta$

$x = 0$ to $x = 1$

$\theta = 0$ to $\theta = \frac{\pi}{4}$




$$\int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2} = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta} \, d\theta = \int_0^{\pi/4} \frac{1}{\sec^2\theta} \, d\theta$$

$$\int_0^{\pi/4} \cos^2\theta \, d\theta = \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/4}$$

$$= \frac{\pi}{8} + \frac{\sin \frac{\pi}{2}}{4} - 0 - 0 = \frac{\pi}{8} + \frac{1}{4}$$

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$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5 \sqrt{9x^2 - 1}} = \int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5 \sqrt{9(x^2 - \frac{1}{9})}}$$

$$= \int_{\sqrt{2}/3}^{2/3} \frac{dx}{3x^5 \sqrt{x^2 - 1/9}}$$

$x^2 - a^2$ $\quad$ $x = a\sec\theta$

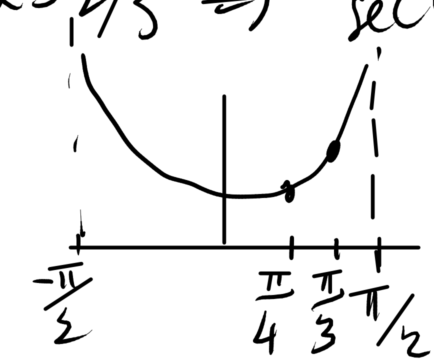
$a = \frac{1}{3}$ $\quad$ $x = \frac{1}{3}\sec\theta$

$dx = \frac{1}{3} \tan\theta \sec\theta \, d\theta$

$x = \frac{1}{3}\sec\theta$

$x = \sqrt{2}/3 \Rightarrow \sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \quad \theta = \frac{\pi}{4}$

$x = 2/3 \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3}$




$$\sqrt{x^2 - \frac{1}{9}} = \sqrt{\left(\frac{1}{3}\sec\theta\right)^2 - \frac{1}{9}}$$

$$= \sqrt{\frac{1}{9}(\sec^2\theta - 1)} = \frac{1}{3}\tan\theta$$

$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{3x^5 \sqrt{x^2 - \frac{1}{9}}} = \int_{\pi/4}^{\pi/3} \frac{\frac{1}{3}\tan\theta \sec\theta \, d\theta}{3\left(\frac{1}{3}\sec\theta\right)^5 \frac{1}{3}\tan\theta}$$

$$\int_{\frac{\pi}{4}}^{\pi/3} 3^4 \sec^{-4}\theta \, d\theta = 3^4 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^4\theta \, d\theta$$

$$\cos^4\theta = (\cos^2\theta)^2 = \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)$$

$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$$

$\int$ get $\frac{1}{4}\left(\theta + \sin 2\theta + \frac{\theta}{2} + \frac{\sin 4\theta}{8}\right)$

$$3^4\left[\frac{1}{4}\left(\theta + \sin 2\theta + \frac{\theta}{2} + \frac{\sin 4\theta}{8}\right)\right]_{\frac{\pi}{4}}^{\pi/3}$$

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$$\int \frac{x \, dx}{\sqrt{3 - 2x - x^2}}$$

complete the square

we can deal with $x^2 + c$

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$$

$$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$$

$$3-2x-x^2=-(x^2+2x-3)$$

$$(x+1)^2=x^2+2x+1$$

$$(x+1)^2-4=x^2+2x-3$$

$$3-2x-x^2=4-(x+1)^2$$

$$\int\frac{x\,dx}{\sqrt{3-2x-x^2}}=\int\frac{x\,dx}{\sqrt{4-(x+1)^2}}$$

sub $\quad x+1=2\sin\theta \Rightarrow \frac{x+1}{2}=\sin\theta$

$x=2\sin\theta-1 \qquad \theta=\sin^{-1}\frac{x+1}{2}$

$$\sqrt{4-(x+1)^2}=\sqrt{4-(2\sin\theta)^2}=\sqrt{4-4\sin^2\theta}$$

$$=\sqrt{4(1-\sin^2\theta)}=\sqrt{4\cos^2\theta}=2\cos\theta$$

$$\left[\begin{array}{ll} u=x+1 & \sqrt{4-(x+1)^2}=\sqrt{4-u^2} \\ u=a\sin\theta & (a=2) \end{array}\right.$$

$x=2\sin\theta-1$

$dx=2\cos\theta\,d\theta$

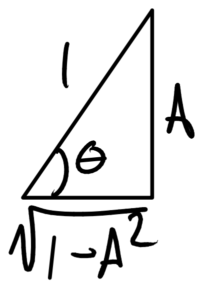
$$\int\frac{x\,dx}{\sqrt{4-(x+1)^2}}=\int\frac{(2\sin\theta-1)\,2\cos\theta\,d\theta}{2\cos\theta}$$

$$\int (2\sin\theta - 1)\, d\theta = -2\cos\theta - \theta + C$$

$$-2\cos\left(\sin^{-1}\left(\frac{x+1}{2}\right)\right) - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$\cos\left(\sin^{-1} A\right)$$

$$= \sqrt{1 - A^2}$$




$$= -2\sqrt{1 - \left(\frac{x+1}{2}\right)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$\int \frac{x\, dx}{\sqrt{x^2 - 2x + 2}} \qquad (x-1)^2 + 1 = x^2 - 2x + 1 + 1$$

$$= \int \frac{x\, dx}{\sqrt{(x-1)^2 + 1}} \qquad \begin{aligned} x - 1 &= \tan\theta \\ dx &= \sec^2\theta\, d\theta \end{aligned}$$

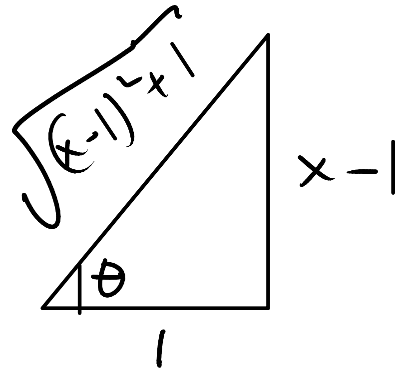
$$= \int \frac{(\tan\theta + 1)\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}}$$

$$= \int (\tan\theta + 1)\sec\theta\, d\theta$$

$$\int \tan\theta \sec\theta \, d\theta + \int \sec\theta \, d\theta$$

$$\sec\theta + \ln\left|\sec\theta + \tan\theta\right| + C$$

$$\theta = \tan^{-1}(x-1)$$




$$\sec\theta = \sqrt{(x-1)^2+1}$$

$$\sqrt{(x-1)^2+1} + \ln\left|\sqrt{(x-1)^2+1} + x-1\right| + C$$