Homework due next Tuesday
There will be class next Tuesday

Integration by parts.

$$\int u\,dv = uv - \int v\,du$$

diff $\quad u = f(x) \qquad v = g(x) \quad$ int.
$\quad du = f'(x)\,dx \qquad dv = g'(x)\,dx$

Integrate once to find v from dv
Integrate again for $\int v\,du$

$$\int \overset{u}{\ln x}\,\overset{dv}{dx} = x \cdot \ln x - \int x \frac{1}{x}\,dx$$

$u = \ln x \qquad v = x$
$du = \frac{1}{x}\,dx \qquad dv = dx$

$$\int \ln x\,dx = x \ln x - \int 1\,dx$$

$$= x \ln x - x + C$$

$$(x \ln x - x)' = (x \ln x)' - 1 = \left(\ln x + x\frac{1}{x}\right) - 1$$
$$= \ln x$$

$$\int t^2 e^t \, dt \qquad u = t^2 \qquad v = e^t$$
$$du = 2t\,dt \qquad dv = e^t\,dt$$

$$\int t^2 e^t \, dt = t^2 e^t - \int e^t \, 2 \cdot t \, dt$$
$$= t^2 e^t - 2\int t e^t \, dt$$

Now compute

$$\int t e^t \, dt = t e^t - \int e^t \, dt = t e^t - e^t$$

$$u = t \qquad v = e^t$$
$$du = dt \qquad dv = e^t \, dt$$

$$\int t^2 e^t = t^2 e^t - 2\left[t e^t - e^t \right] + C$$
$$= t^2 e^t - 2 t e^t + 2 e^t + C$$
$$= \left(t^2 - 2t + 2\right) e^t + C$$

$$\frac{d}{dt}\left[\left(t^2 - 2t + 2\right) e^t\right] = \left(t^2 - 2t + 2\right) e^t + (2t - 2) e^t$$
$$= t^2 e^t \quad \checkmark$$

$$\int x \tan^2 x \, dx$$

$u = \tan^2 x$ $\qquad$ $v = x^2/2$

$du = 2\tan x \sec^2 x \, dx$ $\qquad$ $dv = x\,dx$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cot^2 x}$$

$$\| $$

$$\frac{1 - \cos^2 x}{\cos^2 x} = \sec^2 x - 1$$

$$\frac{\sin^2 x + \cos^2 x = 1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\int x(\sec^2 x - 1)\,dx = \int x \sec^2 x \, dx - \int x \, dx$$

$$= \left(\int x \sec^2 x \, dx\right) - \frac{x^2}{2}$$

$u = x$ $\qquad$ $v = \tan x$

$du = dx$ $\qquad$ $dv = \sec^2 x \, dx$

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$$

$$u = \cos x$$
$$du = -\sin x\,dx$$

$$= -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int x\tan^2 x\,dx = \int x\sec^2 x\,dx - \frac{x^2}{2}$$

$$= x\tan x - \ln|\sec x| - \frac{x^2}{2} + C$$

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

$u = \sin x$ $\qquad$ $v = e^x$

$du = \cos x\,dx$ $\qquad$ $dv = e^x\,dx$

$$= e^x \sin x - \left\{ e^x \cos x - \int e^x(-\sin x)\,dx \right\}$$

$u = \cos x$ $\qquad$ $v = e^x$

$du = -\sin x\,dx$ $\qquad$ $dv = e^x\,dx$

$$\underbrace{\int e^x \sin x\,dx}_{=I} = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

$$2I = e^x \sin x - e^x \cos x$$

$$I = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

Trigonometric integrals

- $\int \sin^n x \cos^m x \, dx$
- $\int \tan^n x \sec^m x \, dx$
- $\int \sin ax \cos bx \, dx$

Some functions can't be integrated eg. $\int e^{-x^2} dx$

Antiderivative is $F(t) = \int_0^t e^{-x^2} dx$

$F(t)$ can't be expressed in terms of polynomials e^x, $\ln x$, $\sin x$, $\sin^{-1} x$, etc.

$\int \sin^n x \cos^m x \, dx$ (n and m are whole numbers)

e.g. $\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$

$\begin{pmatrix} n=0 \\ m=3 \end{pmatrix}$ $\cos^2 x = 1 - \sin^2 x$

$= \int (1 - \sin^2 x) \cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$= \int (1-u^2) \, du = u - \frac{u^3}{3} + C$

$= \sin x - \frac{\sin^3 x}{3} + C$

$\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \sin x \cos^2 x \, dx$

$= \int (1-\cos^2 x)^2 \sin x \cos^2 x \, dx$ because $\sin^4 x = (\sin^2 x)^2$

$u = \cos x$
$du = -\sin x \, dx$

$= \int (1-u^2)^2 u^2 (-du) = -\int (1-2u^2+u^4) u^2 \, du$

$$= -\int (u^2 - 2u^4 + u^6)\,du = -\left(\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right) + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

$$\int \sin^n x \cos^m x \, dx$$

If n — the power of $\sin x$ — is odd, take one factor of $\sin x$ for du, convert rest into $\cos x$ using pythagorean identity

$$\sin^2 x = (1 - \cos^2 x)$$

Reversing the roles of $\sin x$ and $\cos x$, a similar trick works if the power of $\cos x$ is odd.

$$\int \cos^4 x \, dx$$

$\cos^2 x \overset{\text{related}}{\rightsquigarrow} \cos 2x$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx = \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$$

$$= \frac{1}{4}\int \left(1+2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{4}\left[x + \sin 2x + \frac{x}{2} + \frac{1}{8}\sin 4x\right] + C$$

- $\sin 2x = 2\sin x\cos x \implies \sin x\cos x = \frac{1}{2}\sin 2x$
- $\cos 2x = \cos^2 x - \sin^2 x$

$$\cos 2x = (1-\sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$$

$$\cos 2x - 1 = -2\sin^2 x$$

$$\frac{1}{2}(1-\cos 2x) = \sin^2 x$$

$$\cos 2x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$$

$$\cos 2x + 1 = 2\cos^2 x \longrightarrow \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\int \sin 4x \sin 3x \, dx = \int \frac{1}{2}\left[\cos(4x-3x) - \cos(4x+3x)\right] dx$$

$$= \int \frac{1}{2}\left[\cos x - \cos 7x\right] dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 7x}{7}\right] + C$$

① $\cos(A+B) = \cos A \cos B - \sin A \sin B$

② $\cos(A-B) = \cos A \cos(-B) - \sin A \sin(-B)$

$= \cos A \cos B + \sin A \sin B$

③ $\sin(A+B) = \sin A \cos B + \cos A \sin B$

④ $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\sin A \sin B$

2 $-$ 1

$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$

$\frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] = \sin A \sin B$

$\int \tan^n x \sec^m x \, dx$ $\qquad \sec^2 x = 1 + \tan^2 x$

Try $u = \tan x$
$du = \sec^2 x \, dx$

$\Rightarrow$ save a $\sec^2 x$, convert rest to tangent.

works if power of sec x is even

Try $u = \sec x$
$du = \sec x \tan x \, dx$

$\Rightarrow$ Save a $\sec x \tan x$ convert rest to sec x

works if power of tangent is odd.

$\int \tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x \, (\sec x \tan x) \, dx$

$= \int (\sec^2 x - 1)^2 \sec^2 x \, (\sec x \tan x) \, dx$

$u = \sec x \quad du = \sec x \tan x \, dx$

$= \int (u^2 - 1)^2 u^2 \, du$