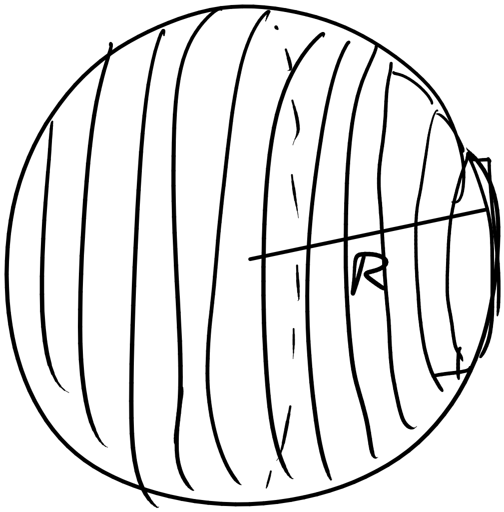
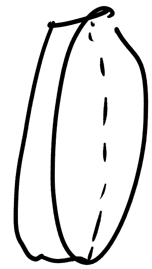
Volumes



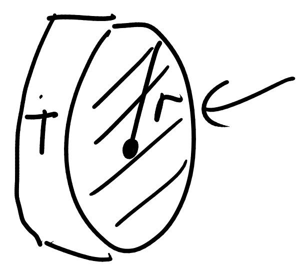


Volume $\approx \sum$ (Areas of disks $\cdot$ thickness)

Volume of a disk

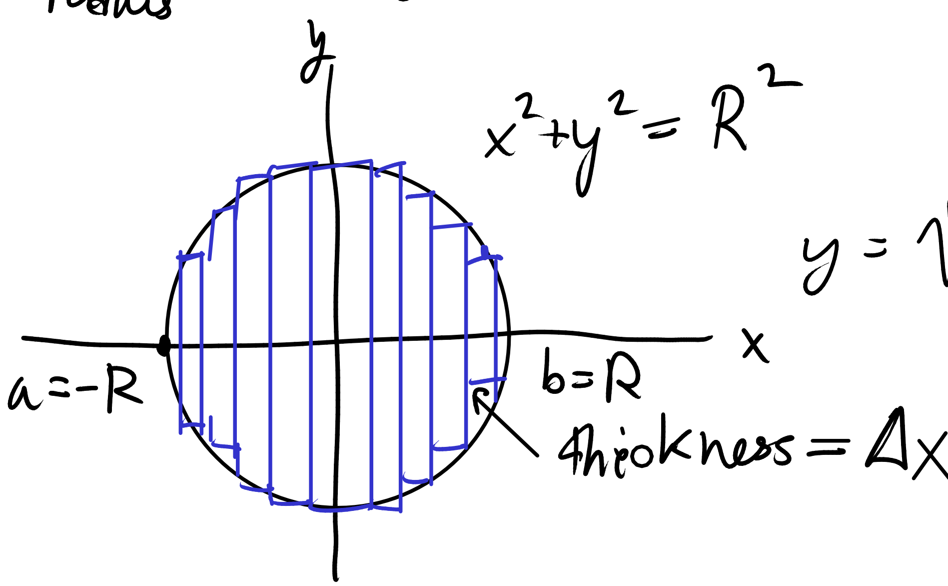
$$= \pi r^2 \cdot T$$

radius thickness



$$x^2+y^2=R^2$$

$$y=\sqrt{R^2-x^2}$$



radius of each disk depends on y value

$$V \approx \sum \pi (y_i^*)^2 \Delta x = \sum \pi f(x_i^*)^2 \Delta x$$

$$V = \int_{-R}^{R} \pi f(x)^2 \, dx$$

$$V = \int_{-R}^{R} \pi \left[\sqrt{R^2 - x^2}\right]^2 dx$$

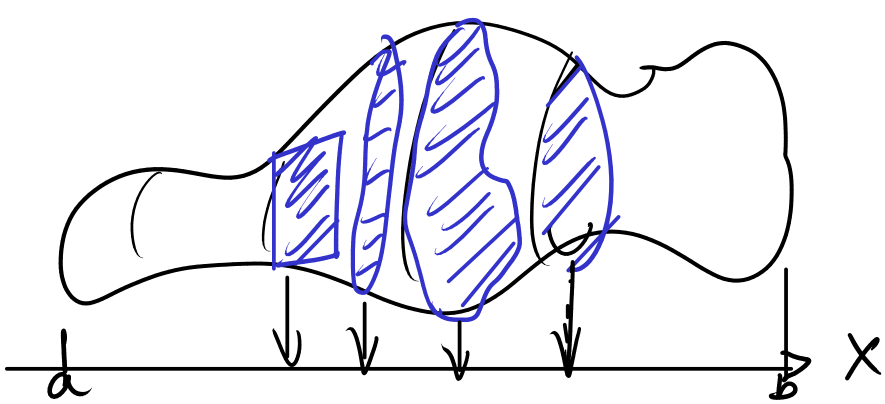
$$= \int_{-R}^{R} \pi (R^2 - x^2) \, dx = \pi \left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R}$$

$$= \pi \left[\left(R^3 - \frac{R^3}{3}\right) - \left(-R^3 + \frac{R^3}{3}\right)\right]$$

$$= \pi \left[R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3}\right] = 2\pi \left(R^3 - \frac{R^3}{3}\right)$$

$$= 2\pi R^3 \left(\frac{2}{3}\right) = \frac{4}{3} \pi R^3$$

In general: Take a 3D shape
slice it into cross sections

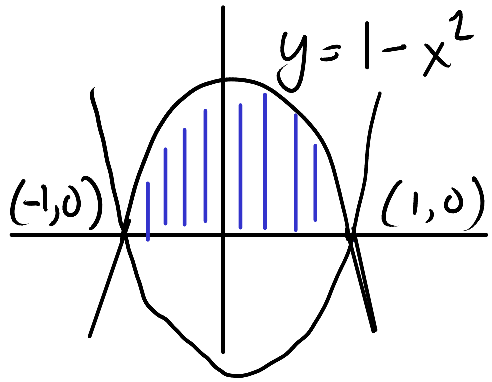


For each x-value, you get a slice
let $A(x)$ denote the area of this slice.

Then the volume is $V = \int_a^b A(x)\,dx$

Total volume = integral of cross-sectional area.

Eg.



revolve around x-axis

it sweeps out a 3D shape

$V = ?$ cross sections are circles

radius $= y = 1 - x^2$

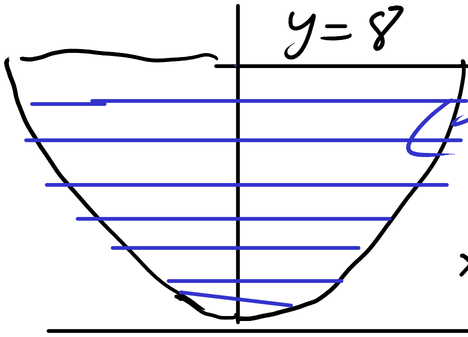
cross sectional area $A(x) = \pi y^2 = \pi(1-x^2)^2$

$$V = \int_{-1}^{1} \pi(1-x^2)^2\,dx = \int_{-1}^{1} \pi\left[1 - 2x^2 + x^4\right]dx$$

$$\parallel$$

$$2\int_0^1 \pi(1-x^2)^2\,dx$$ because even.

6? No.



revolve around y-axis

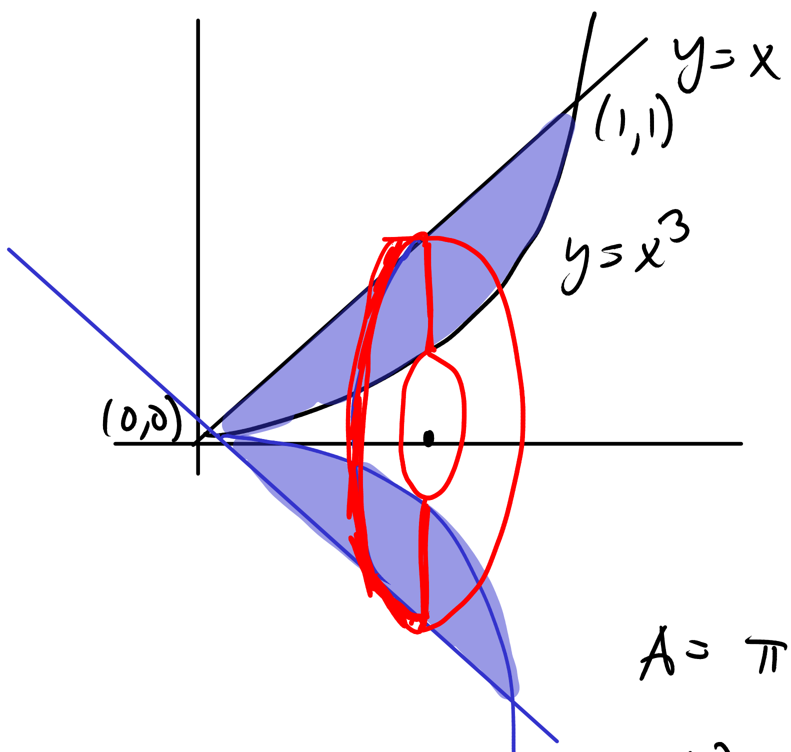
$y = x^3$

$x = y^{1/3}$ use horizontal discs

radius $= x$ Thickness $= \Delta y$

cross sectional area $= \pi x^2 = \pi (y^{1/3})^2 = \pi y^{2/3}$

$$V = \int \pi x^2 \, dy = \int_0^8 \pi y^{2/3} \, dy$$



revolve around x-axis.

cross section is a "washer".

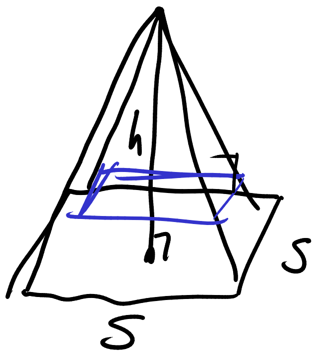
Thickness $= \Delta x$

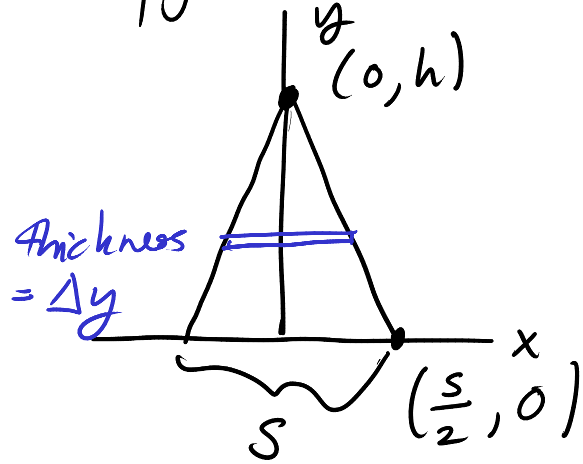
$A = \pi(\text{outer radius})^2 - \pi(\text{inner})^2$

$A(x) = \pi x^2 - \pi (x^3)^2$

$$V = \int A(x)\,dx = \int_0^1 \left[\pi x^2 - \pi (x^3)^2\right] dx$$

find volume of a square pyramid





$$m = \frac{0-h}{\frac{s}{2}-0} = \frac{-h}{s/2} = \frac{-2h}{s}$$

$$y = -\frac{2h}{s}x + h$$

$$y - h = -\frac{2h}{s}x$$

$$\frac{-s}{2h}(y-h) = x$$

$$-\frac{sy}{2h} + \frac{s}{2} = x$$

side length $= 2x = -\frac{sy}{h} + s$

$$A = (2x)^2 = \left(-\frac{sy}{h} + s\right)^2$$

$$V = \int_0^h \left(-\frac{sy}{h} + s\right)^2 dy$$

$$= s^2 \int_0^h \left(1 - \frac{y}{h}\right)^2 dy$$

substitution $u = \frac{y}{h}$ $\quad du = \frac{dy}{h}$ $\quad dy = h\,du$

$$= s^2 \int_0^1 (1-u)^2 h\,du$$

$$= s^2 h \int_0^1 (1-u)^2 \, du$$

$$\int_0^1 (1-u)^2 \, du = \int_0^1 (1 - 2u + u^2) \, du$$

$$= \left[u - u^2 + \frac{u^3}{3} \right]_0^1 = 1 - 1 + \frac{1}{3} - 0 = \frac{1}{3}$$

$$V = \frac{1}{3} s^2 h$$

Integration by parts. (product rule)

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

$$\int \frac{d}{dx}\left(f(x)g(x)\right) dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Integration by parts.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$u = f(x), \quad du = f'(x)\,dx, \quad v = g(x), \quad dv = g'(x)\,dx$$

$$\int u \, dv = uv - \int v \, du$$

Definite integrals

$$\int_a^b f(x)g'(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

$$= f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

$$\int x \sin x \, dx$$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$$\int u\,dv = uv - \int v\,du$$

$$\int x \sin x\,dx = x(-\cos x) - \int -\cos x\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

$$\frac{d}{dx}\left(-x\cos x + \sin x\right) = -\cos x + x\sin x + \cos x$$

$$= x\sin x$$

$\int \ln x \, dx \longleftarrow$ Next time