Substitution Rule

Chain Rule $\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$

Substitution rule

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$

$\frac{du}{dx} = g'(x)$ $\quad du = g'(x)\,dx$

Variable changes $u = g(x)$

differential changes $du = g'(x)\,dx$

E.g. $\int x^3 \cos(x^4+2)\,dx$

Try $u = x^4+2$

$du = 4x^3\,dx \longrightarrow \frac{1}{4}du = x^3\,dx$

$$\int x^3 \cos(x^4+2)\,dx = \int \cos(x^4+2)(x^3\,dx)$$

$$= \int \cos(u)\, \frac{1}{4}\, du = \frac{1}{4} \int \cos u \, du$$

$$= \frac{1}{4} \sin u + C = \frac{1}{4} \sin(x^4+2) + C$$

$$\int x^2 \cos(x^2)\, dx \qquad u = x^2$$
$$du = 2x\, dx$$

not a match.

$$\int x (\cos u)\left(\frac{1}{2} du\right) = \int \sqrt{u} \cos u \, \frac{1}{2} du$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx \qquad \cancel{u = \sin x}$$
$$\cancel{du = \cos x\, dx}$$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$\int \frac{-du}{u} = -\int \frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

$$\int \frac{x}{1+x^4}\,dx \qquad \cancel{\begin{array}{l} u = 1+x^4 \\ du = 4x^3\,dx \end{array}}$$

$$u = x^2 \qquad du = 2x\,dx \qquad x\,dx = \frac{1}{2}\,du$$

$$\int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C$$

$$= \frac{1}{2}\tan^{-1}(x^2) + C$$

Substitution applied to definite integrals:

Need to also transform limits of integration:

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

where $u = g(x)$

$du = g'(x)\,dx$

$g(a)$ = u-value that corresponds to $x=a$

$g(b)$ = " " " " $x=b$

$$\int_0^4 \sqrt{2x+1}\, dx$$

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u}\, \frac{1}{2} du = \frac{1}{2} \int \sqrt{u}\, du$$

$u = 2x+1$

$du = 2\,dx$

$$= \frac{1}{2} \frac{u^{3/2}}{3/2} + C = \frac{1}{2} \frac{2}{3} u^{3/2} + C$$

$$= \frac{1}{3} u^{3/2} + C = \frac{1}{3} (2x+1)^{3/2} + C$$

$$\int_0^4 \sqrt{2x+1}\, dx = \left[\frac{1}{3} (2x+1)^{3/2} \right]_0^4$$

$$= \frac{1}{3} (2 \cdot 4 + 1)^{3/2} - \frac{1}{3} (2 \cdot 0 + 1)^{3/2}$$

$$= \frac{1}{3} 9^{3/2} - \frac{1}{3} 1^{3/2} = \frac{1}{3} (27 - 1) = \frac{26}{3}$$

Again $$\int_0^4 \sqrt{2x+1}\, dx = \int_1^9 \sqrt{u}\, \frac{1}{2} du$$

$u = 2x+1$ $\qquad x = 0 \Rightarrow u = 2 \cdot 0 + 1 = 1$

$du = 2dx$ $\qquad x = 4 \Rightarrow u = 2 \cdot 4 + 1 = 9$

$$\int_1^9 \frac{1}{2} \sqrt{u}\, du = \left[\frac{1}{2} \frac{2}{3} u^{3/2} \right]_1^9 = \left[\frac{1}{3} u^{3/2} \right]_1^9$$

$$= \frac{1}{3} 9^{3/2} - \frac{1}{3} 1^{3/2} = \frac{1}{3}(27-1) = \frac{26}{3}$$

WRONG $$\int_0^4 \sqrt{2x+1}\, dx = \int_0^4 \frac{1}{2}\sqrt{u}\, du$$

$$= \left[\frac{1}{3} u^{3/2}\right]_0^4 = \frac{1}{3} 4^{3/2} = \frac{8}{3}$$ WRONG

$$\int_0^1 \cos \frac{\pi t}{2}\, dt = \int_0^{\pi/2} \cos u \left(\frac{2}{\pi}\right) du$$

$$u = \frac{\pi t}{2} \qquad du = \frac{\pi}{2} dt$$

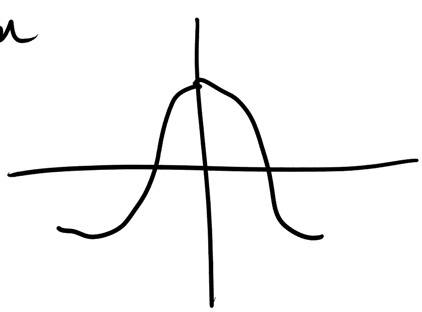
$$\int_0^{\frac{\pi}{2}} \cos u \left(\frac{2}{\pi}\right) du = \frac{2}{\pi} \int_0^{\pi/2} \cos u\, du$$

$$= \frac{2}{\pi}\left[\sin u\right]_0^{\pi/2} = \frac{2}{\pi}(1-0) = \frac{2}{\pi}$$

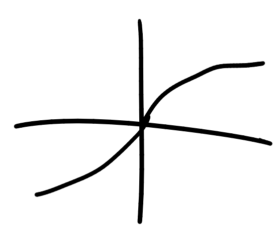
Symmetry of functions   Even function $f(-x)=f(x)$

ODD function $f(-x) = -f(x)$

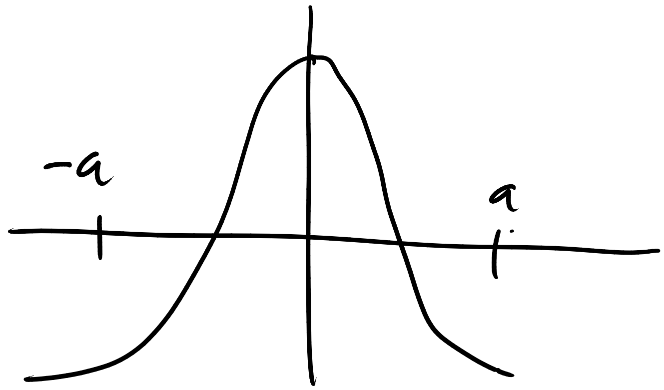
even



odd



Suppose $f(x)$ is even consider $\int_{-a}^{a} f(x)\,dx$



$$\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

Do a substitution on $\int_{-a}^{0} f(x)\,dx$ $\quad u=-x$, $du=-dx$

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)\,(-du) = -\int_{a}^{0} f(-u)\,du$$

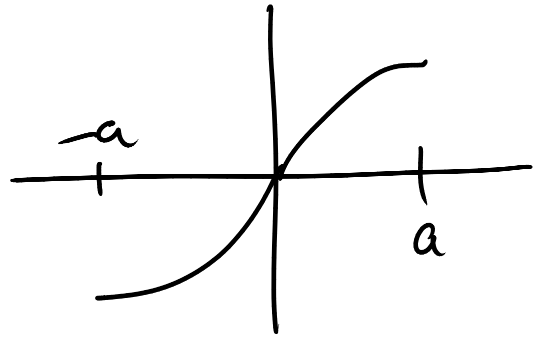
$$= -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx$$

$$\therefore \int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx \quad \text{If } f(x) \text{ even}$$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ even}$$

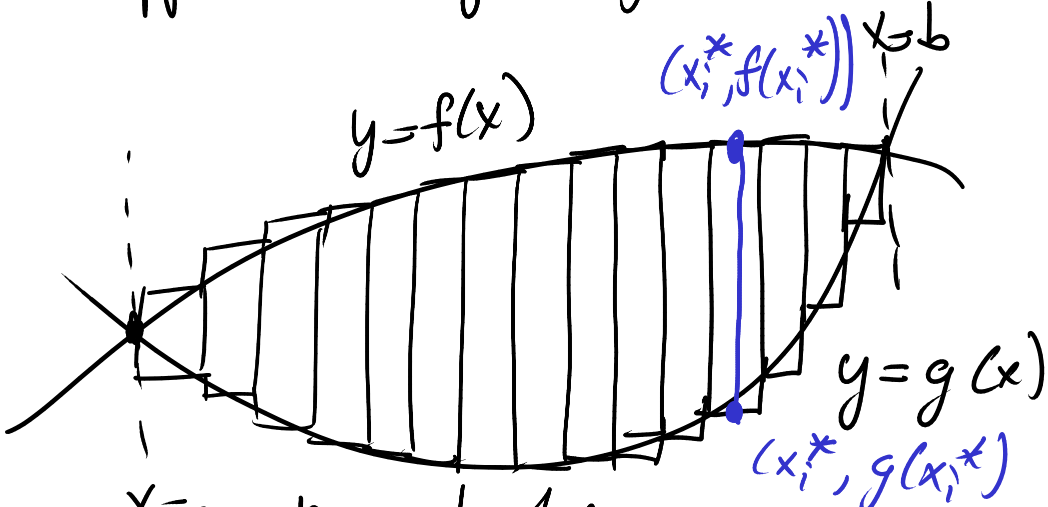
If $f(x)$ odd



Then $\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$

$\therefore \int_{-a}^{a} f(x)\,dx = 0$

Application of integration: area between curves.



n rectangles

width $= \Delta x = \dfrac{b-a}{n}$

height $= f(x_i^*) - g(x_i^*)$ $\qquad x_i^*$ in the ith interval

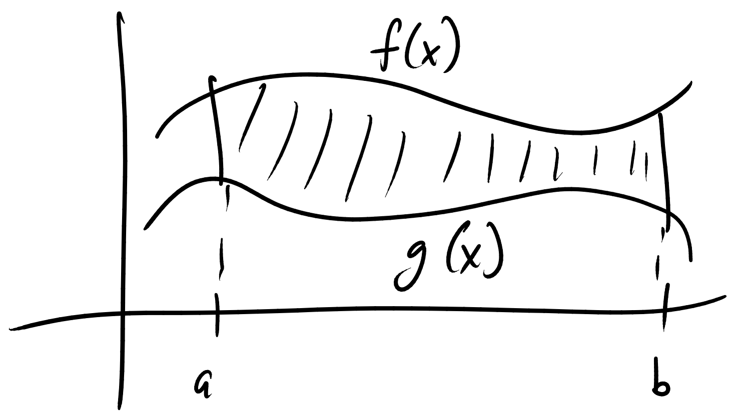
Area $\approx \displaystyle\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \Delta x$$

$$= \int_a^b \left[f(x) - g(x) \right] dx$$

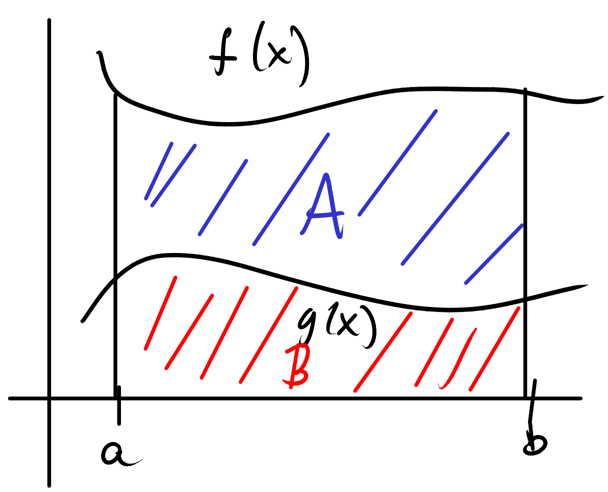
$$\text{“} \int_{\text{left}}^{\text{right}} [\text{top} - \text{bottom}]\, dx \text{”}$$

This is valid assuming $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$



$$A = \int_a^b [f(x) - g(x)]\, dx$$

Assume $f(x) \geq g(x) \geq 0$

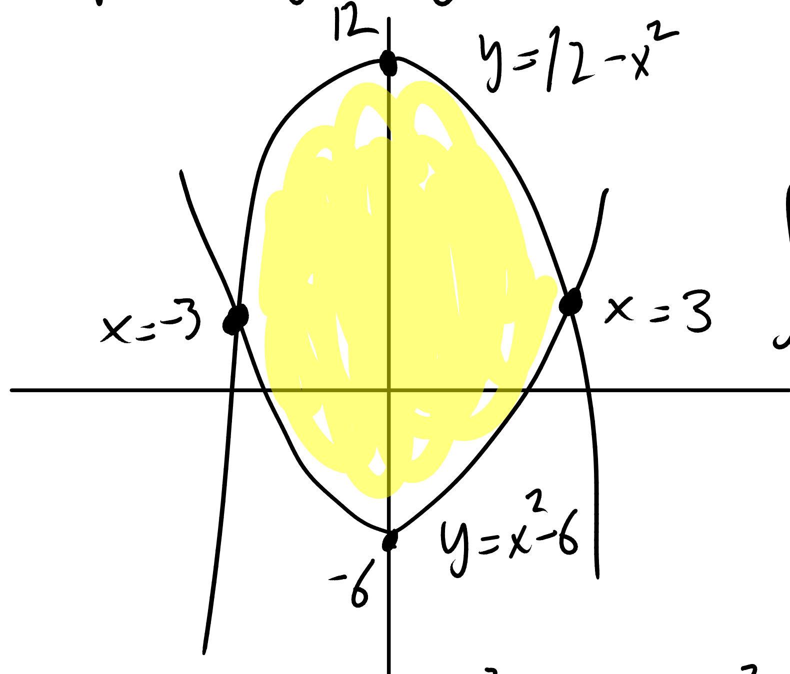


$$A + B = \int_a^b f(x)\, dx$$

$$B = \int_a^b g(x)\, dx$$

$$A = (A+B) - B = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$
$$= \int_a^b [f(x) - g(x)]\,dx$$

Find the area of the region enclosed by the graphs of $y = 12 - x^2$ and $y = x^2 - 6$.



$$\int_{-3}^{3} [12 - x^2 - (x^2 - 6)]\,dx$$

Find corners: $12 - x^2 = y = x^2 - 6$

$$12 - x^2 = x^2 - 6$$
$$18 = 2x^2$$
$$9 = x^2$$
$$\pm 3 = x$$

$a = -3$
$b = 3$

$$A = \int_{-3}^{3} [12 - x^2 - (x^2 - 6)]\,dx$$

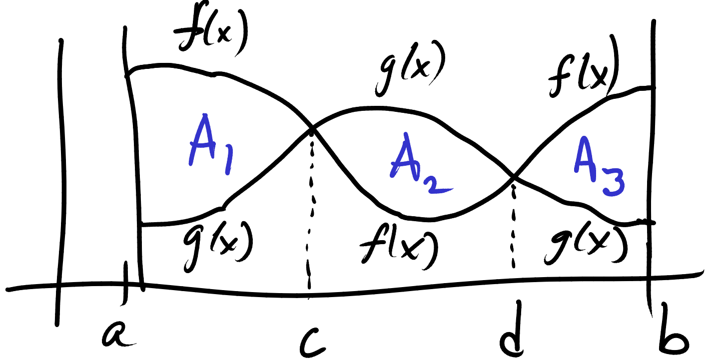
$$= \int_{-3}^{3} \left[18 - 2x^2 \right] dx = 2 \int_0^3 \left[18 - 2x^2 \right] dx$$

because integrating even function from $-a$ to a.

$$= 2 \left[18x - \frac{2}{3} x^3 \right]_0^3 = 2 \left[18 \cdot 3 - \frac{2}{3} (3)^3 - 0 \right]$$

$$= 2 \left[18 \cdot 3 - 2 \cdot 9 \right] = 2 \cdot 9 \left[2 \cdot 3 - 2 \right] = 2 \cdot 9 \cdot 4$$

$$= 72$$



Area between curves $= A_1 + A_2 + A_3$

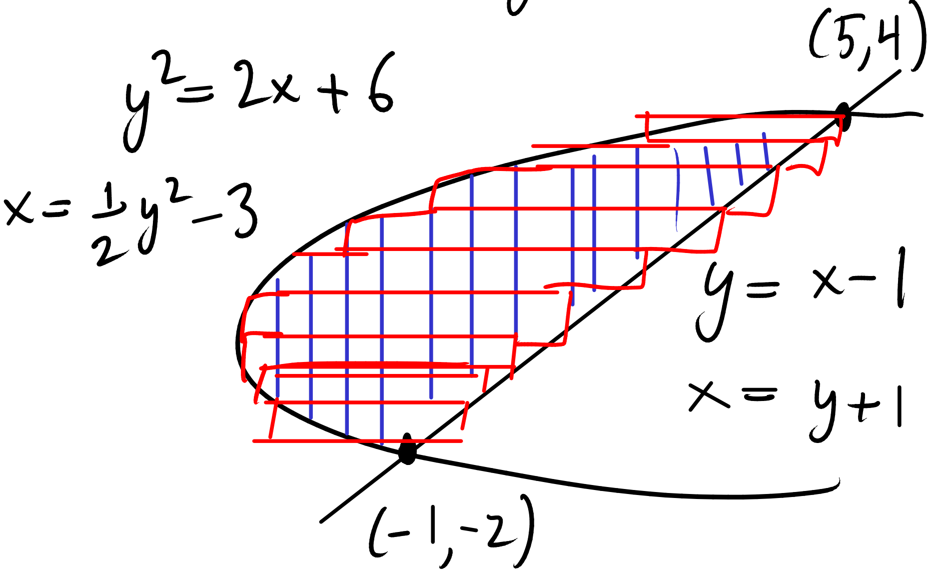
$$A_1 = \int_a^c \left[f(x) - g(x) \right] dx \qquad A_3 = \int_d^b \left[f(x) - g(x) \right] dx$$

$$A_2 = \int_c^d \left[g(x) - f(x) \right] dx$$

Always want to integrate a positive function.

Same as $\displaystyle\int_a^b |f(x) - g(x)|\,dx = A_1 + A_2 + A_3$

Can also integrate with respect to y



$$\int_{\text{bottom}}^{\text{top}} [\text{right} - \text{left}]\,dy$$

$$\int_{-2}^{4} \left[(y+1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy = 18$$