Indefinite integrals: New homework posted
Due next Tuesday

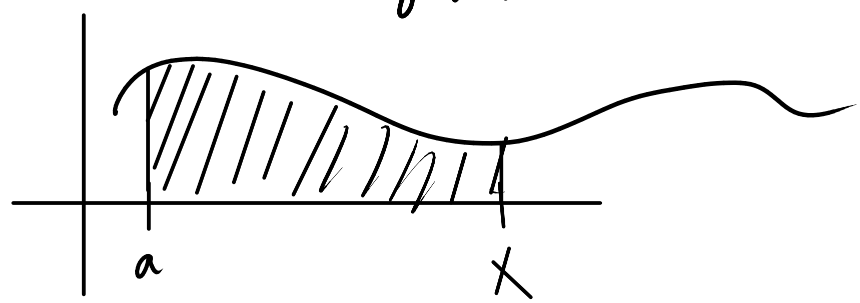
Recall Fundamental Theorem of Calculus

FTC 1: $g(x) = \int_a^x f(t)\,dt$

Then $g'(x) = f(x)$ $\qquad f \xrightarrow{\text{integrate}} g \xrightarrow{\text{diff}} f$

Derivative of g is f, g is an antiderivative of f.

$\int_a^x f(t)\,dt$



$\int_a^x f(x)\,dx$ ← considered bad, using x for two diff things.

Dummy variable $\int_a^x f(t)\,dt = \int_a^x f(s)\,ds = \int_a^x f(y)\,dy$

FTC 2: $\int_a^b f(x)\,dx = F(b) - F(a)$

where F is an antiderivative of f

$F'(x) = f(x)$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$F \xrightarrow{\text{diff}} F' \xrightarrow{\text{integrate}} F(b) - F(a)$$

Indefinite integral = antiderivative

$$\int x^2\,dx = \frac{x^3}{3} + C \quad \Big| \quad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

Indefinite integral is a function (with an undetermined constant)

definite integral $\int_a^b f(x)\,dx$ is number.

FTC 2: $$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

where $$\left[F(x)\right]_a^b = F(b) - F(a)$$

Eg: $$\int_3^6 \frac{dx}{x}$$

$$\int \frac{dx}{x} = \int \frac{1}{x}\,dx = \int x^{-1}\,dx$$

$$= \ln|x| + C$$

$$\int_3^6 \frac{dx}{x} = \Big[\ln|x| + C\Big]_3^6 = \ln|6| + C - \ln|3| - C$$

$$= \ln|6| - \ln|3| = \ln 6 - \ln 3$$

$$= \ln\frac{6}{3} = \ln 2$$

Indefinite integral rules

$$\int c f(x)\,dx = c\int f(x)\,dx$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C \qquad \int k\,dx = k\int 1\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C \qquad \int \csc x \cot x \, dx = -\csc x + C$$

$$\int \frac{1}{x^2+1} \, dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$$

Is the following student true?

$$\int \frac{1}{x^2\sqrt{1+x^2}} \, dx = -\frac{\sqrt{1+x^2}}{x} + C$$

$$\frac{d}{dx}\left(-\frac{\sqrt{1+x^2}}{x}\right) = -\frac{x\left[\frac{1}{2}(1+x^2)^{-1/2} 2x\right] - \sqrt{1+x^2}}{x^2}$$

$$= \frac{-x^2(1+x^2)^{-1/2} + \sqrt{1+x^2}}{x^2}$$

$$= \frac{(1+x^2)^{-1/2}\left(-x^2 + (1+x^2)^1\right)}{x^2}$$

$$= \frac{(1+x^2)^{-1/2}(1)}{x^2} = \frac{1}{x^2\sqrt{1+x^2}} \quad \checkmark$$

$$\int (x^2 + x^{-2})\,dx = \int x^2\,dx + \int x^{-2}\,dx$$

$$= \frac{x^3}{3} + \frac{x^{-2+1}}{-2+1} + C = \frac{x^3}{3} + \frac{x^{-1}}{-1} + C$$

$$= \frac{x^3}{3} - \frac{1}{x} + C$$

$$\int (\csc^2 t - 2e^t)\,dt = \int \csc^2 t\,dt - 2\int e^t\,dt$$

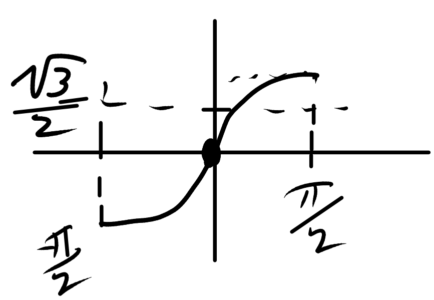
$$= -\cot t - 2e^t + C$$

We can absorb all constants of integration into a single undetermined constant in the final answer.

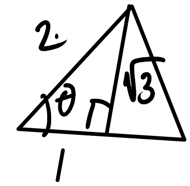
$$\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1-r^2}} \qquad \int \frac{dr}{\sqrt{1-r^2}} = \sin^{-1} r + C$$

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{dr}{\sqrt{1-r^2}} \overset{\text{FTC2}}{=\!=} \left[\sin^{-1} r\right]_0^{\sqrt{3}/2}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0$$



$$\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$$



$$\sin^{-1}0=0$$

$$\int_0^{\sqrt{3}/2}\frac{dr}{\sqrt{1-r^2}}=\sin^{-1}\frac{\sqrt{3}}{2}-\sin^{-1}0=\frac{\pi}{3}-0=\frac{\pi}{3}$$

$$\int_0^{\pi/4}\frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta=\int_0^{\frac{\pi}{4}}\left(\frac{1}{\cos^2\theta}+1\right)d\theta$$

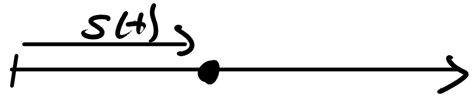
$$=\int_0^{\pi/4}\left(\sec^2\theta+1\right)d\theta=\Big[\tan\theta+\theta\Big]_0^{\pi/4}$$

$$=\tan\frac{\pi}{4}+\frac{\pi}{4}-(\tan 0+0)$$

$$=1+\frac{\pi}{4}-(0+0)=\frac{\pi}{4}+1$$

Interpretation of FTC2:

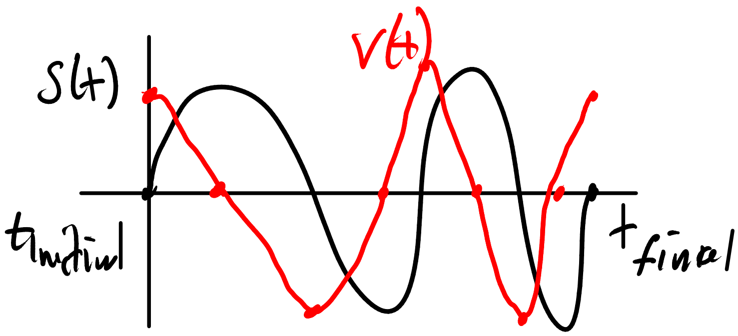
$$\underbrace{\int_a^b \underbrace{F'(x)}_{\substack{\text{rate}\\\text{of change}}}\,dx}_{\substack{\text{integral of}\\\text{rate of change over }[a,b]}}=\underbrace{F(b)-F(a)}_{\substack{\text{net change in }F\\\text{over the interval }[a,b]}}$$



$s(t)$ = position $\qquad v(t) = s'(t)$

$v(t)$ = velocity

$$\int_{t_{initial}}^{t_{final}} v(t)\,dt = s(t_{final}) - s(t_{initial})$$



← in this example net change = 0.

$\int_{t_{initial}}^{t_{final}} v\,dt = 0$ as well.

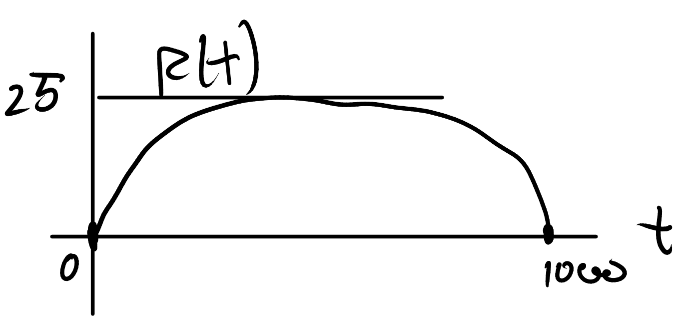
Storm: Rain falls into a lake, no outflow.

Rate of rainfall: $\quad R(t) = \frac{1}{10000}\, t(1000 - t)$

in m^3/s $\qquad$ ← rate

Storm starts at $t=0$ and ends at $t=1000$ s

← limits of integration.



What is net change in total volume of water in the lake?

FTC 2: Net change $= \int (\text{rate})\,dt$

$$\int_0^{1000} \frac{1}{10000} t(1000-t)\,dt$$

$$= \int_0^{1000} \frac{1}{10000}(1000t - t^2)\,dt$$

$$= \int_0^{1000} \left(\frac{t}{10} - \frac{t^2}{10000}\right) dt$$

$$= \left(\frac{t^2}{20} - \frac{t^3}{30000}\right]_0^{1000} = \frac{(1000)^2}{20} - \frac{(1000)^3}{30000} - (0-0)$$

$$= \frac{(10^3)^2}{2\cdot 10} - \frac{(10^3)^3}{3\cdot 10^4} = \frac{10^6}{2\cdot 10} - \frac{10^9}{3\cdot 10^4} =$$

$$= \frac{10^5}{2} - \frac{10^5}{3} = 10^5\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}10^5 \text{ m}^3$$

Substitution (u-substitution)

(equivalent to the chain rule)

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\int 2x \sqrt{1+x^2}\, dx \qquad u = 1+x^2 = g(x)$$

$$f'(u) = \sqrt{u} \qquad g'(x) = 2x$$

$$f'(g(x)) = \sqrt{1+x^2}$$

$$2x\sqrt{1+x^2} = f'(g(x))\, g'(x) = \frac{d}{dx}\left[f(g(x))\right]$$

$$f'(u) = \sqrt{u} = x^{1/2} \Rightarrow f(u) = \frac{2}{3}u^{3/2}$$

$$f(g(x)) = f(1+x^2) = \frac{2}{3}(1+x^2)^{3/2}$$

Check: $\frac{d}{dx}\left[\frac{2}{3}(1+x^2)^{3/2}\right] = \frac{2}{3} \cdot \frac{3}{2}(1+x^2)^{1/2} \cdot 2x$

$$= 2x\left(1+x^2\right)^{1/2} = 2x\sqrt{1+x^2} \quad \checkmark$$

$$\int 2x\sqrt{1+x^2}\,dx = \frac{2}{3}\left(1+x^2\right)^{3/2} + C$$

$$\int 2x\sqrt{1+x^2}\,dx$$

try $u = 1+x^2$ $\qquad du = 2x\,dx$

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}\left(1+x^2\right)^{3/2} + C$$