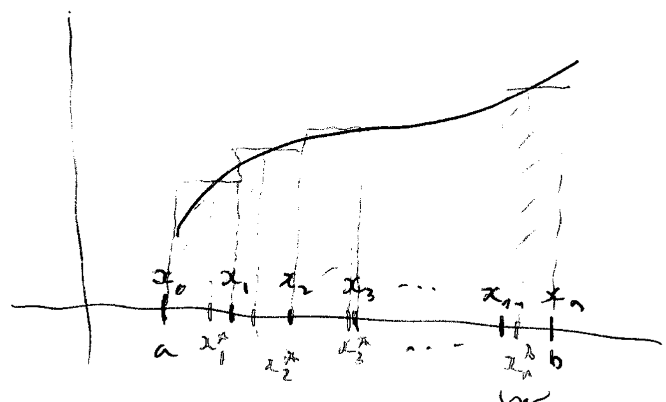
Dr Pascaleff is away.

I am Dr Gunningham

Reminder on integrals:

Let $f(x)$ be a function on $[a, b]$

subdivide $[a,b]$ into n equal parts and choose sample points $x_i^* \in [x_{i-1}, x_i]$



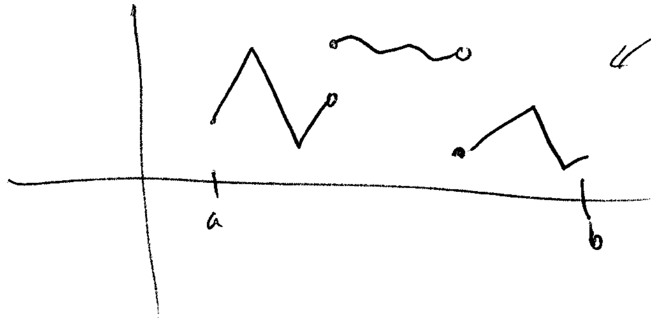
$$\Delta x = \frac{b-a}{n}$$

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x\right) = \int_a^b f(x)\,dx$$

provided that the limit exists and is equal for all choices of sample points.

<u>Thm</u>: If $f(x)$ is continuous on $[a,b]$ (or if it has a finite number of jump discontinuities), then $f(x)$ is integrable on $[a,b]$.

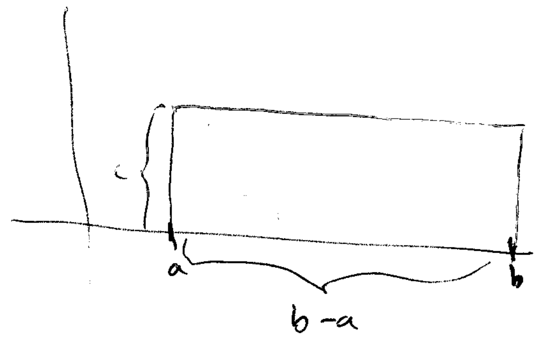
e.g



integrable.

Properties of the integral:

① $\displaystyle\int_a^b c \, dx = c \cdot (b-a)$



② $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

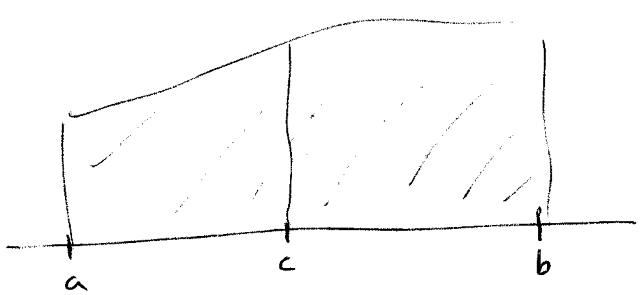
③ $\displaystyle\int_a^a f(x)\,dx = 0$

④ $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

⑤ $\displaystyle\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$

⑥ $\displaystyle\int_a^b f(x)\,dx$

$\displaystyle= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



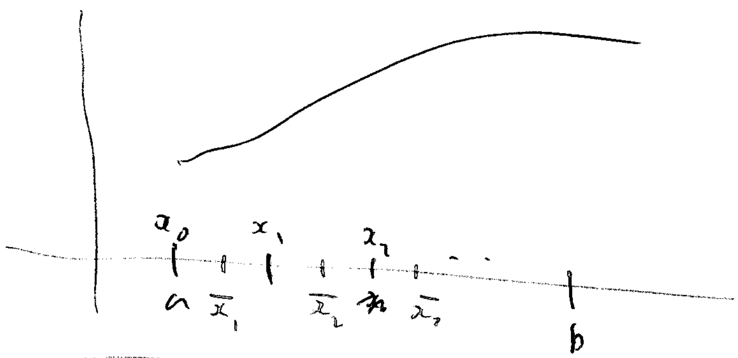
7. If $f(x) \leq g(x)$ for all $x \in [a,b]$,

then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

<u>Midpoint rule</u>: We can approximate the integral

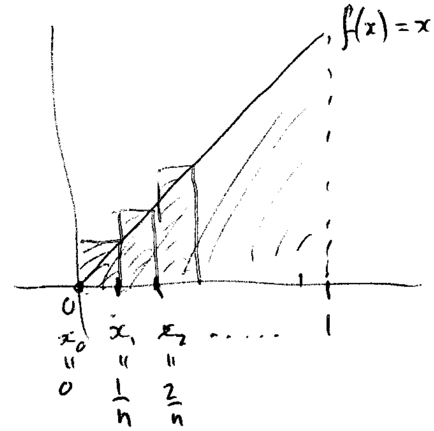
$\int_a^b f(x)\,dx$ by computing a Riemann sum

using the midpoints of each subinterval as sample points.



$$\boxed{\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\cdot \Delta x}$$

Lets compute $\int_0^1 x\,dx = \frac{1}{2}$



Divide $[0,1]$ into n equal parts.

$$x_i = \frac{i}{n}$$

choose $x_i^* = x_i = \frac{i}{n}$

$$\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)\cdot\left(\frac{1}{n}\right)$$

$$= \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2}$$

$$= \frac{1}{n^2}\left(1 + 2 + 3 + \cdots + n\right)$$

$$\begin{array}{r} 1 + 2 + 3 + \cdots + n \\ + \; n + (n-1) + (n-2) + \cdots + 1 \end{array} = 2 \times (1 + 2 + 3 + \cdots + n)$$

$$= n(n+1)$$

$$\Rightarrow \quad 1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1)$$

$$\Rightarrow \int_0^1 x\,dx = \lim_{n\to\infty} \frac{1}{n^2}\cdot\left(\frac{1}{2} n(n+1)\right) = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}$$

Fundamental Theorem of Calculus

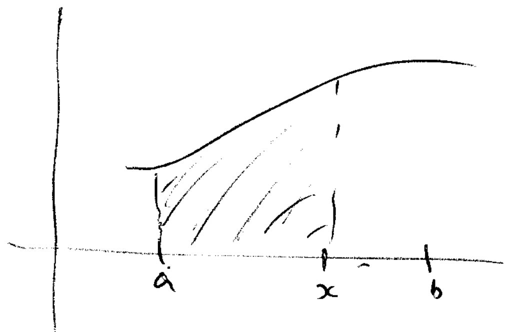
Let f be a continuous function on $[a,b]$

Consider the function

$$g(x) = \int_a^x f(t)\,dt$$

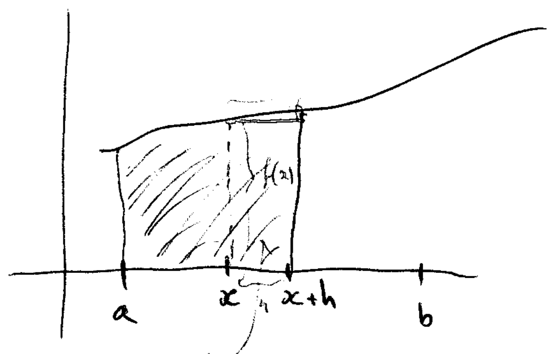
Note: $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(\smile)\,d\smile$$

$$\boxed{g'(x) = f(x)}$$



Why is this:

$$\frac{g(x+h)-g(x)}{h}$$



$$\int_x^{x+h} f(t)\,dt = g(x+h) - g(x) \approx h \cdot f(x)$$

$$\rightsquigarrow \underbrace{\frac{g(x+h)-g(x)}{h}}_{\approx\, g'(x)} \approx \frac{1}{h}\left(h \cdot f(x)\right) = f(x)$$

In other words $\underline{g \text{ is an antiderivative of } f}$

i.e. $g'(x) = f(x)$

F.T.C. Version 2: If ~~~~ $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b$$

If $F(x)$ is any anti-derivative of $f(x)$, then

$F(x) + C$ is also an antiderivative.

e.g. if $f(x) = x^2$, $F(x) = \frac{1}{3}x^3$

Suppose $G(x)$ is another antiderivative of $f(x)$,

$$\frac{d}{dx}\Big(F(x) - G(x)\Big) = F'(x) - G'(x) = 0$$

MVT $\Rightarrow$ $F(x) - G(x) = C$ $\rightsquigarrow$ $F(x) = G(x) + C$.

Proof: Note that $g(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

$$\Rightarrow \quad F(x) - g(x) = C \quad , \text{ constant}$$

plug in a: $F(a) - \cancel{g(a)}^{0} = C \quad \Rightarrow \quad F(a) = C$

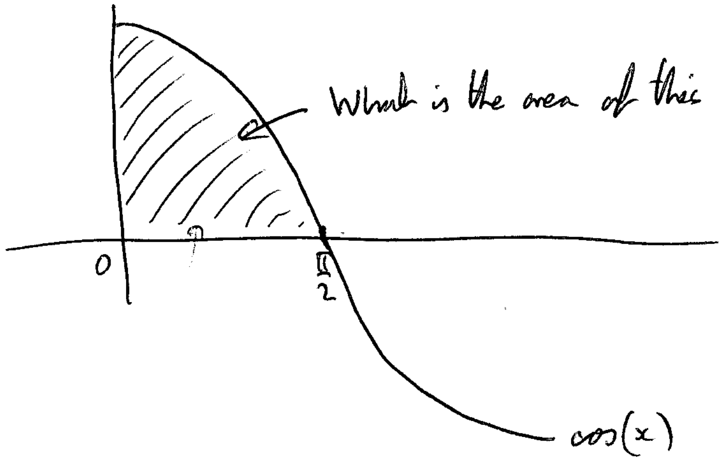
$$\Rightarrow \quad F(x) - g(x) = F(a)$$

plug in b: $F(b) - \int_a^b f(t)\,dt = F(a)$

$$F(b) - F(a) = \int_a^b f(t)\,dt$$

//

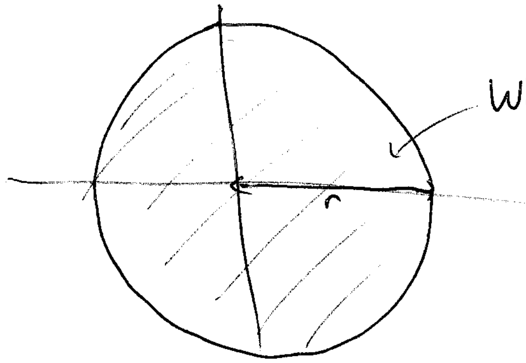
e.g. Compute



$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = F\left(\frac{\pi}{2}\right) - F(0)$$ for any antiderivative $F(x)$.

Take $F(x) = \sin(x)$.

$$= \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$$

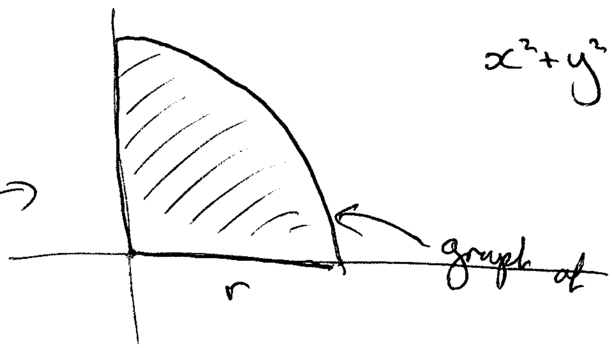


What is the area of this?

$\boxed{\pi r^2}$

Proof:

Compute area of



graph of $f(x)=\sqrt{r^2-x^2}$

$x^2+y^2=r^2 \rightsquigarrow y=\sqrt{r^2-x^2}$

$$\mathrm{Area}(\ ) = \int_0^r \sqrt{r^2-x^2}\,dx$$

Claim: $F(x) = \dfrac{r^2}{2}\sin^{-1}\left(\dfrac{x}{r}\right) + \dfrac{x}{2}\sqrt{r^2-x^2}$

Check: $F'(x) = \dfrac{r^{\cancel{2}}}{2}\dfrac{1}{\sqrt{1-\frac{x^2}{r^2}}} + \dfrac{1}{2}\sqrt{r^2-x^2} + \dfrac{x}{2}\cdot\dfrac{\cancel{2}\,-x}{\sqrt{r^2-x^2}}$

$$= \frac{r^2}{2}\frac{1}{\sqrt{(r^2-x^2)}} + \frac{1}{2}\sqrt{r^2-x^2} - \frac{x^2}{2\sqrt{r^2-x^2}}$$

$$= \frac{1}{2}\left(\frac{r^2-x^2}{\sqrt{r^2-x^2}} + \frac{r^2-x^2}{\sqrt{r^2-x^2}}\right) ???$$

$$\rightsquigarrow F(r) - F(0)$$

$$\parallel$$

Area () $= \frac{\pi^2 r^2}{4}$