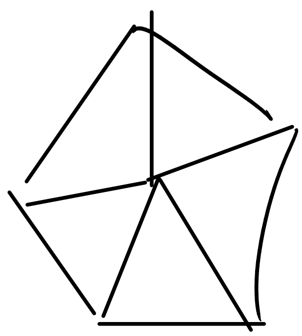
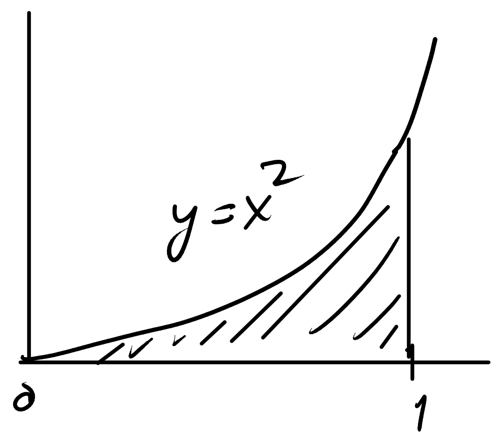
HW due next Tues at 2:00pm.

Riemann sums and definite integrals

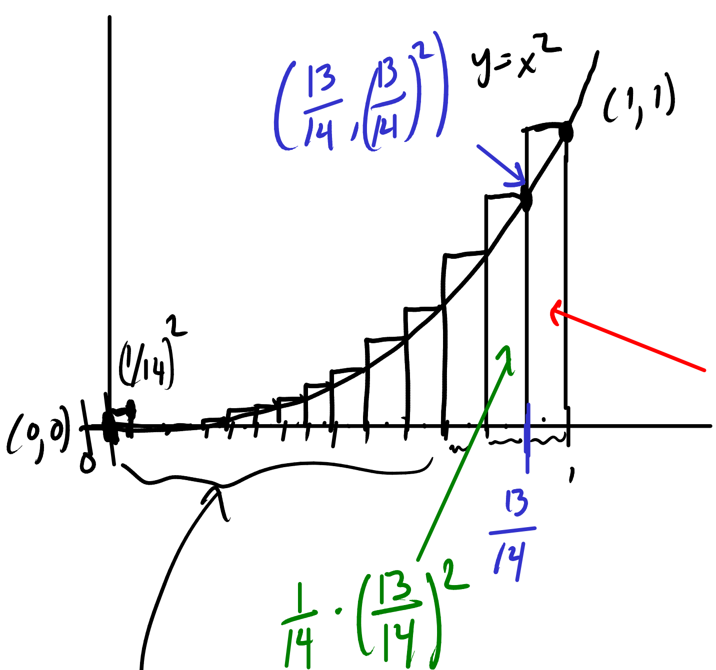
About finding areas.





Find area under the curve.

Approximate answer break area up into rectangles



14 intervals

width of each rectangle $\frac{1}{14}$

$A = 1\cdot\frac{1}{14}$

these x coordinates are

$$0,\ \frac{1}{14},\ \frac{2}{14},\ \frac{3}{14},\ \ldots,\ \frac{13}{14},\ \frac{14}{14}=1$$

heights: $\cancel{0^2}, \left(\frac{1}{14}\right)^2, \left(\frac{2}{14}\right)^2, \ldots, \left(\frac{13}{14}\right)^2, 1^2$

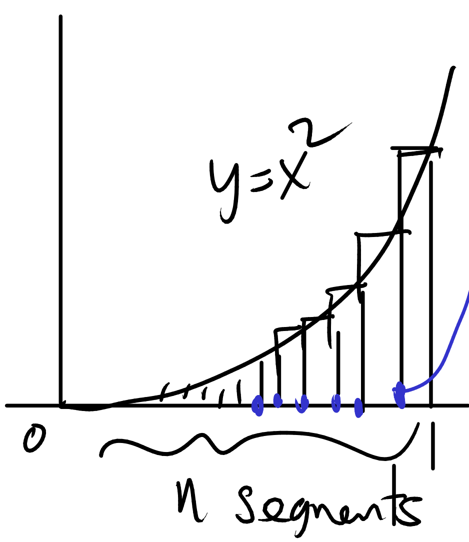
widths: $\frac{1}{14}, \frac{1}{14}, \frac{1}{14}, \ldots, \frac{1}{14}, \frac{1}{14}$

Areas of rectangles $\frac{1}{14}\left(\frac{1}{14}\right)^2, \frac{1}{14}\left(\frac{2}{14}\right)^2, \frac{1}{14}\left(\frac{3}{14}\right)^2, \ldots, \frac{1}{14}\left(\frac{13}{14}\right)^2, \frac{1}{14}1^2$

Total: $\underbrace{\frac{1}{14}\left(\frac{1}{14}\right)^2 + \frac{1}{14}\left(\frac{2}{14}\right)^2 + \frac{1}{14}\left(\frac{3}{14}\right)^2 + \cdots + \frac{1}{14}\left(\frac{13}{14}\right) + \frac{1}{14}1^2}_{\text{Sigma notation}}$

$$\| $$

$$\sum_{i=1}^{14} \frac{1}{14}\left(\frac{i}{14}\right)^2$$



width $= \frac{1}{n}$

x values

$0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1$

heights $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \ldots, \left(\frac{(n-1)}{n}\right)^2, \left(\frac{n}{n}\right)^2$

Area $\frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2 + \frac{1}{n}\left(\frac{n}{n}\right)^2$

$$\sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$$

This is an approximation but we can get the exact area if we take the limit as $n \to \infty$.

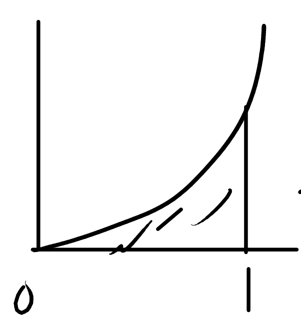
$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^2\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{n^3}\, i^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{i=1}^{n} i^2\right) = \lim_{n\to\infty}\left(\frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right)$$

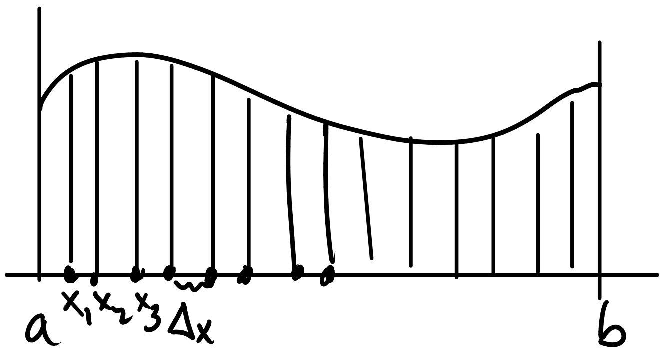
Fact $\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6} = \frac{1\cdot 1\cdot 2}{6} = \frac{1}{3}$$

Exact Area :



$= \frac{1}{3}$



use n intervals of equal width Δx

$$n\,\Delta x = b-a$$

$$\Delta x = \frac{b-a}{n}$$

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

$$x_i = a + i\Delta x = a + \frac{i}{n}(b-a)$$

Heights of rectangles.

Rightend point rule: $f(x_1), f(x_2), f(x_3), \ldots, f(x_n)$

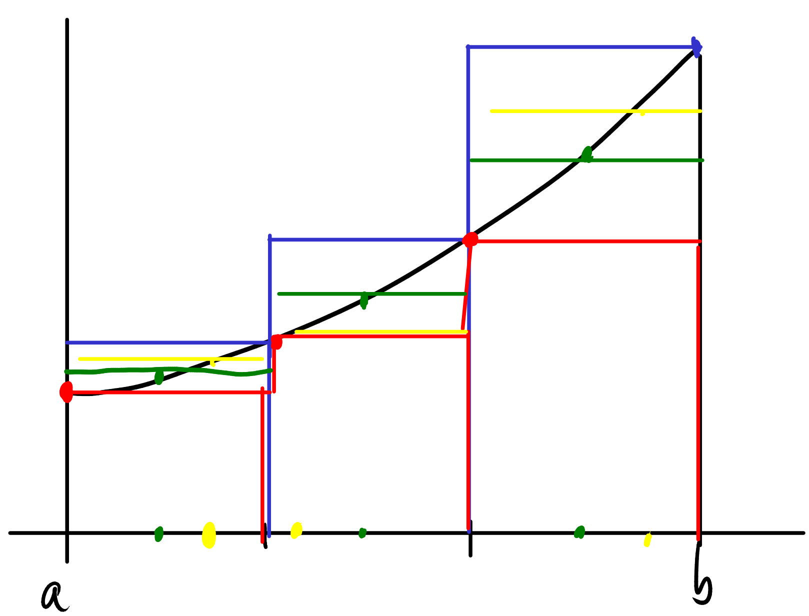
Leftend point rule: $f(x_0), f(x_1), f(x_2), \ldots, f(x_{n-1})$

Midpoint rule $f\left(\frac{x_0+x_1}{2}\right), f\left(\frac{x_1+x_2}{2}\right), \ldots f\left(\frac{x_{n-1}+x_n}{2}\right)$

Right endpoint in blue

Left

midpoint



More generally can use any "sampling scheme"

height of ith rectangle $= f(x_i^*)$

where x_i^* is some point in $[x_{i-1}, x_i]$

Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$

$$= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$$

<u>Definition</u> The definite integral of f on $[a,b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right)$$

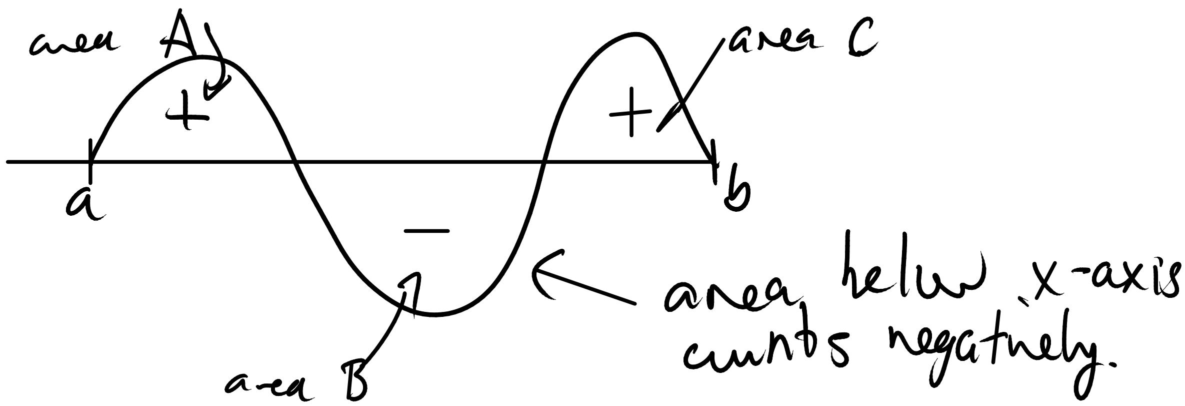
provided that this limit exists.

A function f is called integrable if the limit of the Riemann sums does not depend on the choices made in setting up the Riemann sum.

limits of integration

fancy S $\longrightarrow \displaystyle\int_a^b \boxed{f(x)\,dx}$ integrand

$\int_a^b f(x)\,dx$ is the signed area between the x-axis and the graph of f.



$$\int_a^b f(x)\,dx = A - B + C$$

$\boxed{\text{Thm}}$ A continuous function is integrable.

So at a theoretical level, integration is easier than differentiation.

Practically, the opposite is true.

Consider $\lim_{n\to\infty} \sum_{i=1}^{n} x_i \ln(1+x_i^2)\,\Delta x$

$f(x_i) = x_i \ln(1+x_i^2)$

$[a,b] = [2,6]$ $\qquad \Delta x = \frac{6-2}{n} = \frac{4}{n}$

$$x_i = 2 + i\frac{4}{n}$$

This equals $\int_2^6 x \ln(1+x^2)\,dx$

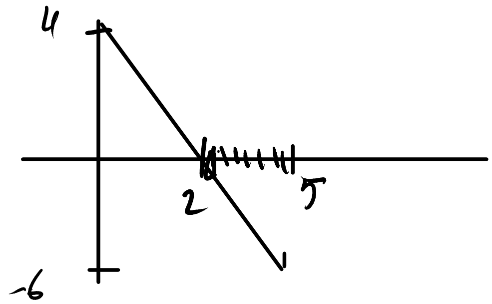
Set up Riemann sum for $\int_2^5 (4-2x)\,dx$

using (n) subdivisions right end point rule

$f(x)=4-2x$

$a=2$
$b=5$

$\Delta x = \frac{5-2}{n} = \frac{3}{n}$



$x_0 = 2 \quad x_1 = 2+\frac{3}{n} \quad x_2 = 2+2\cdot\frac{3}{n} \ldots$

$x_i = 2 + i\frac{3}{n}$

$$f\left(2+\frac{3}{n}\right)\Delta x + f\left(2+2\cdot\frac{3}{n}\right)\Delta x + \cdots$$

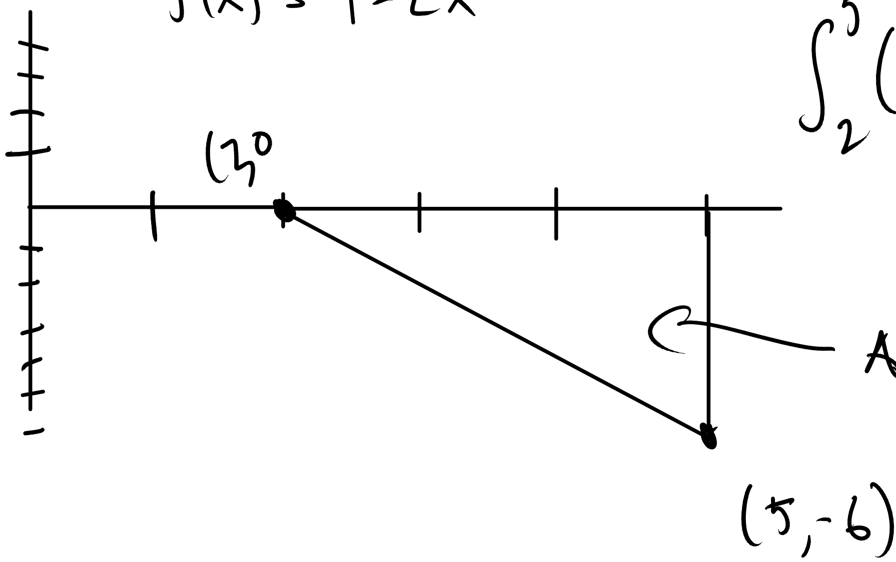
$$\sum_{i=1}^{n} f\left(2+i\frac{3}{n}\right)\frac{3}{n}$$

$$\sum_{i=1}^{n} \left(4-2\left(2+i\frac{3}{n}\right)\right)\frac{3}{n}$$ its been set up

Can take limit as $n\to\infty$, if you use $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

On the other hand, can find $\int_2^5 (4-2x)\,dx$ by looking at it as an area.

$f(x) = 4 - 2x$



$$\int_2^5 (4-2x)\,dx = -9$$

negative because its below x-axis

$$\sum_{i=1}^{n} \left(4 - 2\left(2 + i\frac{3}{n}\right)\right)\frac{3}{n} = \sum_{i=1}^{n} \left(4 - 4 - \frac{i}{n}6\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n} -\frac{i}{n}6\left(\frac{3}{n}\right) = \frac{-18}{n^2}\sum_{i=1}^{n} i = \frac{-18}{n^2}\,\frac{n(n+1)}{2}$$

$$= -9\,\frac{n(n+1)}{n^2} = -9\,\frac{n+1}{n} = -9\left(1 + \frac{1}{n}\right)$$

Take limit as $n \to \infty$

$$\lim_{n\to\infty} -9\left(1 + \frac{1}{n}\right) = -9$$

Proof $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\begin{array}{r} 1 + 2 + \cdots + n \\ n + n-1 + \cdots + 1 \\ \hline n+1 + n+1 + \cdots + n+1 \end{array} \quad \begin{array}{l} \left.\right\} 2\displaystyle\sum_{i=1}^{n} i \\ \\ \left.\right\} n(n+1) \end{array}$$