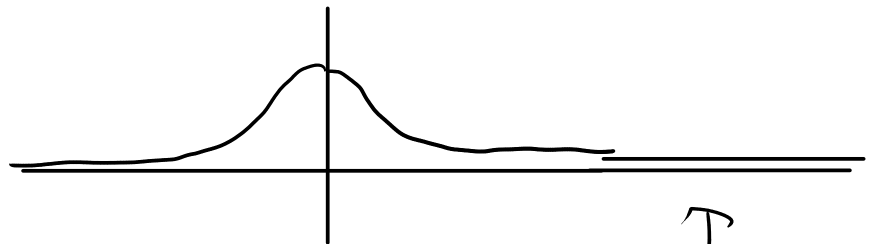
Indeterminate forms and L'Hopital's Rule

Part 0: limits at ∞ $\quad \lim_{x\to\infty} f(x)$

Related to horizontal asymptotes

$$f(x) = \frac{1}{x^2+1}$$



$$\lim_{x\to\infty} \frac{1}{x^2+1} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^2+1} = 0$$

graph approaches
x-axis
$\{y=0\}$

$$\lim_{x\to\infty} \frac{1}{x^2} \cdot \frac{1}{1+\frac{1}{x^2}}$$

$$0 \cdot \frac{1}{1+0} = 0$$

Fact: if $r>0$ is a rational number

then $\boxed{\lim_{x\to\infty} \frac{1}{x^r} = 0}$

eg $\frac{1}{x^3}$ or $\frac{1}{x^{1/2}}$ or $\frac{1}{x^{3/5}}$

If $r > 0$ is a rational number with odd denominator

Then x^r makes sense for negative x,

and $\lim_{x \to -\infty} \frac{1}{x^r} = 0$

$\frac{1}{x^{1/2}}$ $\frac{1}{x^{3/5}}$ $\frac{1}{x^n}$

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{\cancel{x^2}}{\cancel{x^2}} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}$$

$$\lim_{x \to \infty} \frac{3x^2 + 1}{5x^3 + 1} = \lim_{x \to \infty} \frac{x^2}{x^3} \frac{3 + \frac{1}{x^2}}{5 + \frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{1}{x} \frac{3 + \frac{1}{x^2}}{5 + \frac{1}{x^3}} = 0 \cdot \frac{3 + 0}{5 + 0} = 0$$

Part 1:

L'Hopital's Rule.

$\lim_{x\to a} \frac{f(x)}{g(x)}$ would like to say this $= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

That's true if $\lim_{x\to a} g(x) \neq 0$

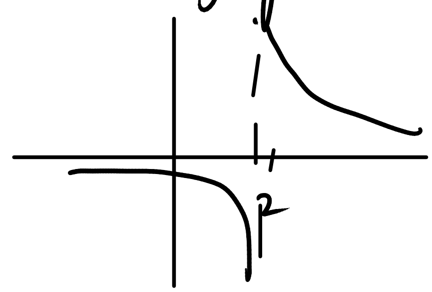
What if denominator $= 0$?

If $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x) \neq 0$

Limit is not a finite number

Typically you have a vertical asymptote

Eg. $\lim_{x\to 2} \frac{1}{x-2}$



What if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$?

This is called the indeterminate form $\frac{0}{0}$

Actual answer could be anything!

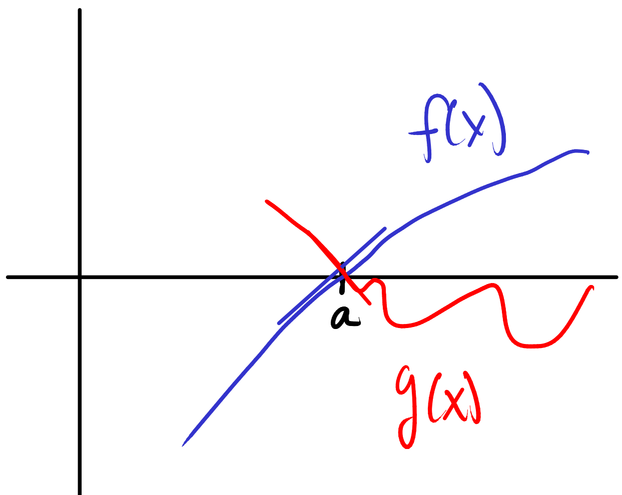
$0, 1, 5, \infty, -\infty$, DNE $\rightarrow$ must do more work.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$$

L'Hopital's rule relates

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{To} \quad \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{in } \frac{0}{0} \text{ case}$$

Reason:



Approximate both functions by their tangent lines.

Assume $f(x), g(x), f'(x), g'(x)$ are continuous.

$$f(x) \approx f(a) + f'(a)(x-a) = f'(a)(x-a)$$
$$g(x) \approx g(a) + g'(a)(x-a) = g'(a)(x-a)$$

Because $f(a) = 0$ and $g(a) = 0$

Because $\lim_{x\to a} f(x) = 0$ and $f(x)$ continuous.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

works as long as $g'(a) \neq 0$

L'Hopital's Rule

Suppose f and g are differentiable
$g'(x) \neq 0$ on some open interval containing a,
except possibly at a itself.

$\frac{0}{0}$: IF $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

THEN $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

$\frac{\infty}{\infty}$ IF $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

THEN $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0} \quad \text{use L'Hopital}$$

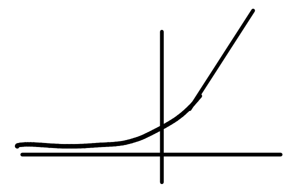
$$\frac{\sin x}{x} \overset{L'H}{\leadsto} \frac{\cos x}{1}$$

$$\lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1 \quad \therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Infinite limits

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$\lim_{x\to\infty} e^x = \infty$$



$$\lim_{x\to\infty} x^2 = \infty \qquad \frac{\infty}{\infty}$$

$$\frac{e^x}{x^2} \leadsto \frac{e^x}{2x}$$

$$\lim_{x\to\infty} \frac{e^x}{2x} \to \frac{\infty}{\infty}$$

$$\frac{e^x}{2x} \leadsto \frac{e^x}{2}$$

$$\lim_{x\to\infty} \frac{e^x}{2} = \infty \Rightarrow \lim_{x\to\infty} \frac{e^x}{2x} = \infty \Rightarrow \lim_{x\to\infty} \frac{e^x}{x^2} = \infty$$

In fact $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ Proof use L'Hopital n times.

"e^x grows faster than any power of x"

Example: $\lim_{x\to 1} \frac{\ln x}{x-1}$ $\frac{0}{0}$

$$\lim_{x\to 1} \frac{\frac{1}{x}}{1} = \frac{\frac{1}{1}}{1} = 1$$

$$\lim_{x\to 2} \frac{\ln x}{x-1} = \frac{\ln 2}{2-1} = \frac{\ln 2}{1}$$

$$\lim_{x\to 2} \frac{\frac{1}{x}}{1} = \frac{1}{2}$$ L'Hopital does not apply

Other indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$

$0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞

Strategy cheat sheet

$$0\cdot\infty = \frac{\infty}{1/0} = \frac{\infty}{\infty} \xrightarrow{L'H} \cdots$$

$$0\cdot\infty = \frac{0}{1/\infty} = \frac{0}{0} \xrightarrow{L'H} \cdots$$

$\infty - \infty \longrightarrow$ common denominator $\longrightarrow \frac{0}{0}$ or $\frac{\infty}{\infty}$

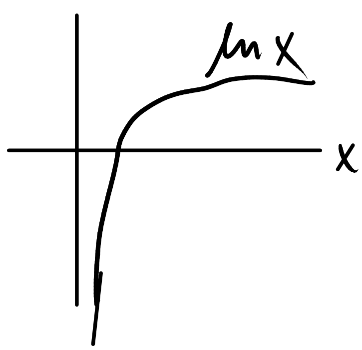
$0^0 \longrightarrow \ln(0^0) = 0 \cdot \ln 0 = 0 \cdot \infty \longrightarrow$ previous case

$\infty^0 \longrightarrow \ln(\infty^0) = 0 \cdot \ln \infty = 0 \cdot \infty \rightarrow$ previous

$1^\infty \longrightarrow \ln(1^\infty) = \infty \cdot \ln(1) = \infty \cdot 0 \rightarrow$ previous

in these case, need to take exponential at the end.

$0 \cdot \infty$ $\quad \lim_{x \to 0^+} \sin x \ln x$ $\quad \lim_{x \to 0^+} \sin x = \sin 0 = 0$



$$\lim_{x \to 0^+} \ln x = -\infty$$

$\lim_{x \to 0^+} \sin x \ln x$ is the indeterminate for $0 \cdot \infty$

$$\sin x \ln x = \frac{\ln x}{(\sin x)^{-1}}$$

$\lim_{x \to 0^+} \frac{\ln x}{(\sin x)^{-1}}$ is $\frac{\infty}{\infty}$

use L'Hopital $\dfrac{\ln x}{(\sin x)^{-1}} \rightsquigarrow \dfrac{\frac{1}{x}}{-(\sin x)^{-2}\cos x}$

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-(\sin x)^{-2}\cos x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x}$$

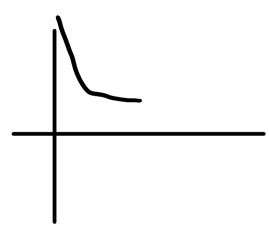
$$= \lim_{x\to 0^+} \frac{-\sin x}{x} \frac{\sin x}{\cos x} = -\lim_{x\to 0^+} \frac{\sin x}{x} \lim_{x\to 0^+} \frac{\sin x}{\cos x}$$

$$= -1 \cdot 0 = 0$$

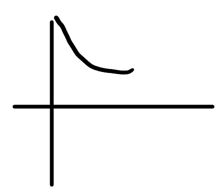
$$\Rightarrow \lim_{x\to 0^+} \frac{\ln x}{(\sin x)^{-1}} = 0 \Rightarrow \lim_{x\to 0^+} \sin x \ln x = 0$$

$$\lim_{x\to 0^+} (\csc x - \cot x)$$

$$\lim_{x\to 0^+} \csc x = \lim_{x\to 0^+} \frac{1}{\sin x} = +\infty$$



$$\lim_{x\to 0^+} \cot x = \lim_{x\to 0^+} \frac{\cos x}{\sin x} = \infty$$



We see $\infty - \infty$

$$\csc x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1-\cos x}{\sin x}$$

$$\lim_{x\to 0^+} \frac{1-\cos x}{\sin x} \to \frac{0}{0}$$

$$\lim_{x\to 0^+} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

$$\therefore \lim_{x\to 0^+} (\csc x - \cot x) = 0$$

The two infinities balance!

<u>Powers:</u> $\lim_{x\to 0^+} x^x \to 0^0$

$$\ln(x^x) = x \ln x$$

$$\lim_{x\to 0^+} x \ln x \to 0 \cdot \infty$$

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \to \frac{\infty}{\infty} \overset{L'H}{\rightsquigarrow} \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} \frac{-x^2}{x}$$

$$= \lim_{x\to 0^+} -x = 0$$

$$\lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = 0 \implies \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = 0$$

$$\implies \lim_{x\to 0^+} x \ln x = 0$$

Q: What is $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}$

$= e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1$

$$\boxed{\lim_{x\to 0^+} x^x = 1}$$

"If you take ln at the beginning, you need to take exponential at the end."