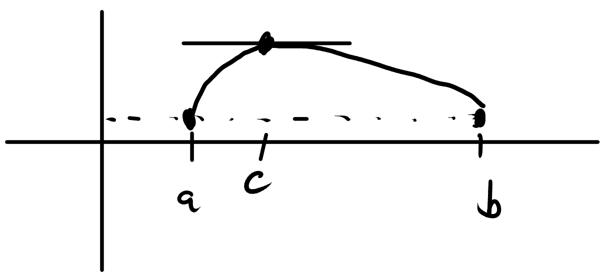
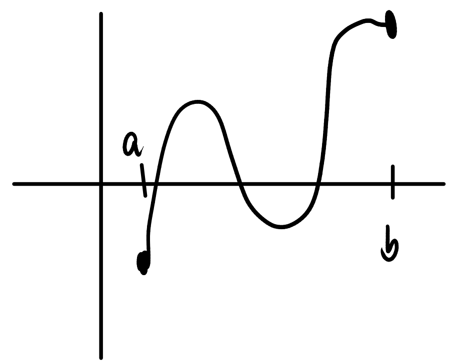
Mean value theorem, derivative tests for max/min.

Recall Rolle's theorem

<u>Theorem</u>: IF f is continuous on $[a,b]$
and f is differentiable on (a,b)
and $f(a)=f(b)$

THEN there is some c in (a,b) such that
$$f'(c)=0$$





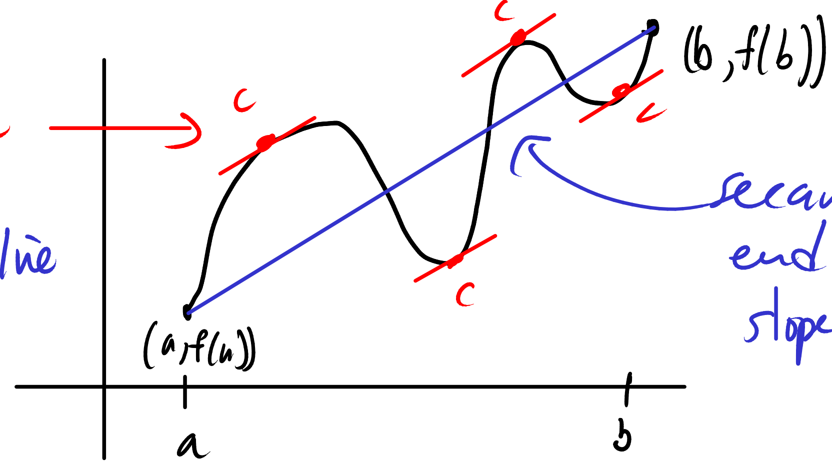
<u>Mean value theorem</u>

IF f is continuous on $[a,b]$
and f is differentiable (a,b)

THEN there is some c in (a,b) such that

$$\underbrace{f'(c)}_{\text{slope of tangent line at point } c.} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of the secant line between } (a,f(a)) \text{ and } (b,f(b))}$$

tangent has same slope to blue line



secant line between end points

slope $= \dfrac{f(b)-f(a)}{b-a}$

$$\underbrace{f'(c)}_{\text{instantaneous rate of change}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{average/mean rate of change over the interval } [a,b]}$$

Proof of MVT from Rolle's Theorem

$$f(x) \qquad g(x) = \underbrace{f(a) + \frac{f(b)-f(a)}{b-a}(x-a)}_{\text{equation of secant line from } (a,f(a)) \text{ to } (b,f(b))}$$

$$h(x) = f(x) - g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

h is continuous on $[a,b]$

h is differentiable on (a,b)

$$h(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0$$

h satisfies the hypotheses of Rolle's theorem

So it satisfies the conclusion: there is c in (a,b) such that

$$h'(c) = 0$$

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

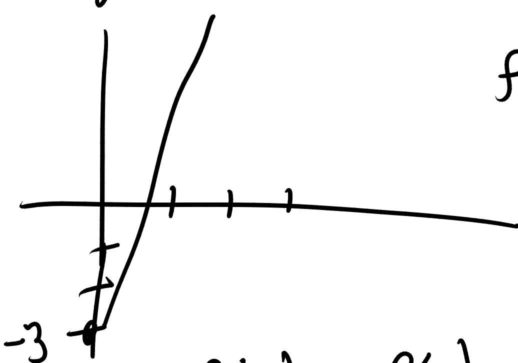
$$h'(c) = 0 \iff 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$\iff f'(c) = \frac{f(b)-f(a)}{b-a}$$

Ex: Know: $f(0) = -3$, $f'(x) \leq 5$

What is the biggest that $f(3)$ could be?

To get biggest $f(3)$ we would need $f'(x) = 5$



$f(3) = -3 + 5 \cdot 3 = 12$

Using MVT: $\frac{f(3)-f(0)}{3-0} = f'(c)$ for some c in $(0,3)$

$$\frac{f(3)-f(0)}{3-0} \le 5$$

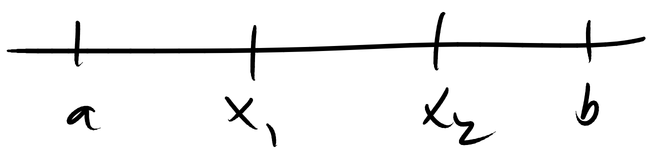
$$\frac{f(3)-(-3)}{3} \le 5$$

$$f(3)-(-3) \le 15$$

$$f(3) \le -3+15 = 12$$

Another application: If $f'(x)=0$ for all x in (a,b) then $f(x)$ is constant.

<u>Proof</u> Take x_1 and x_2 in the interval (a,b)



$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \text{ for some } c \text{ in } (x_1,x_2)$$

$f'(c) = 0$ by assumption.

$$\Longrightarrow f(x_2)-f(x_1)=0 \Rightarrow f(x_1)=f(x_2)$$

The values at any two points are the same $\Rightarrow f$ is constant.

Corollary: if $f'(x) = g'(x)$ for all x in (a,b)

then $f(x) - g(x)$ is constant

$$f(x) = g(x) + c$$

Proof Suppose $f'(x) = g'(x)$ then $(f(x) - g(x))' = f'(x) - g'(x) = 0$

so $f(x) - g(x)$ is constant

call this constant c : $f(x) - g(x) = c$

$$f(x) = g(x) + c$$

Application: Proof that $\sin^2 x + \cos^2 x = 1$

Define $f(x) = \sin^2 x + \cos^2 x$

$$f'(x) = 2\sin x \cos x + 2\cos x(-\sin x)$$
$$= 2\sin x \cos x - 2\sin x \cos x = 0$$

Implies $f(x)$ is constant $\sin^2 x + \cos^2 x = C$

Evaluate at $x = 0$ $\sin 0 = 0$ $\cos 0 = 1$

$f(0) = 0^2 + 1^2 = 1 \Rightarrow \sin^2 x + \cos^2 x = 1$
for every value of x

Consequence of MVT:

→ If $f'(x) > 0$ on an interval, then f is increasing on that interval

If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

f is increasing: For any two points $x_1 < x_2$ we have $f(x_1) < f(x_2)$

f is decreasing: For any two points $x_1 < x_2$ we have $f(x_1) > f(x_2)$

Suppose $f'(x) > 0$ take $x_1 < x_2$

Apply MVT: there is c such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

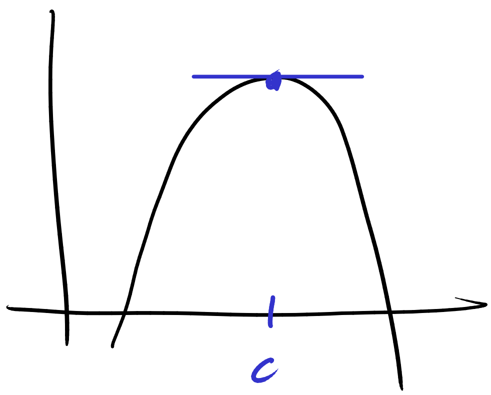
$$f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1) > 0$$

both factors > 0

$$f(x_2) > f(x_1)$$

Last time: If is differentiable, its maxima and minima must occur at points where

$$f'(c) = 0 \leftarrow c \text{ is called a critical number.}$$

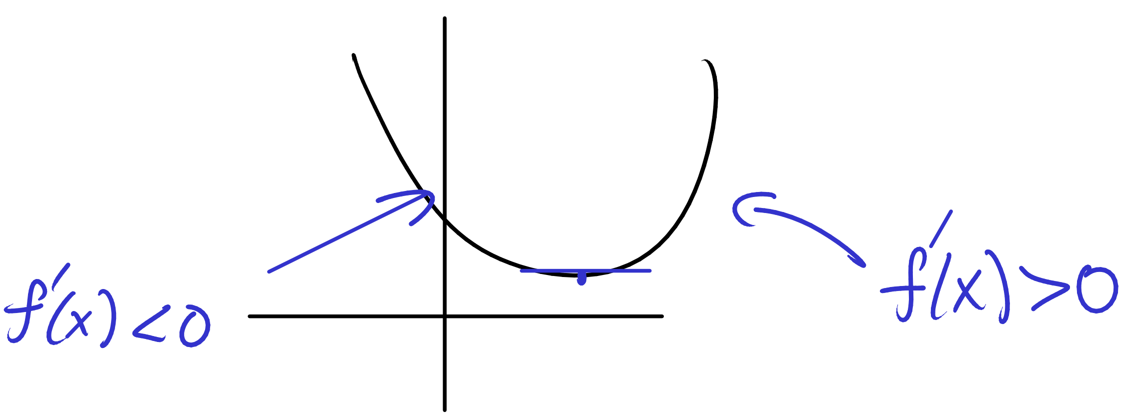


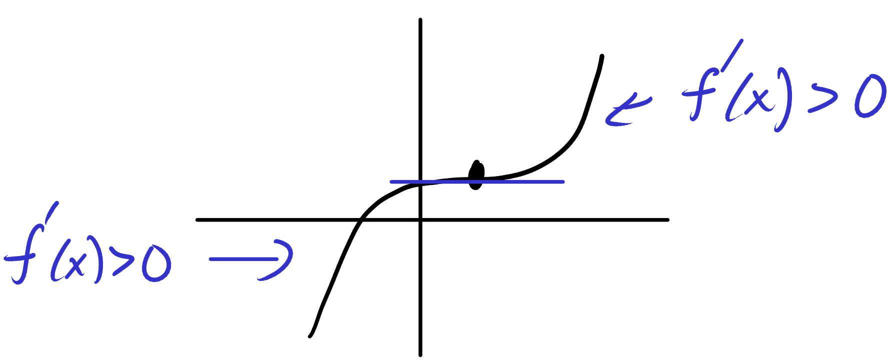
Max $\Leftarrow$ increasing to the left of c and decreasing to the right of c

$\Uparrow$

To the left of c $f'(x) > 0$ and to the right of c $f'(x) < 0$.

For minimum





Neither max nor min.

$f(x) = 2x^3 + 3x^2 - 36x$

find intervals where f is increasing or decreasing
find critical points, classify each as max/min/neither.

$f'(x) = 6x^2 + 6x - 36$

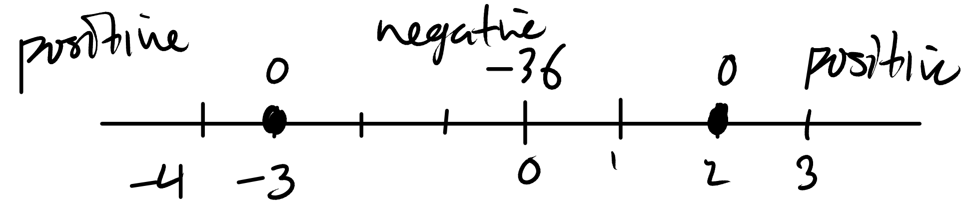
Find critical points $\quad 6x^2 + 6x - 36 = 0$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

critical points $\quad x = -3$ or $x = 2$

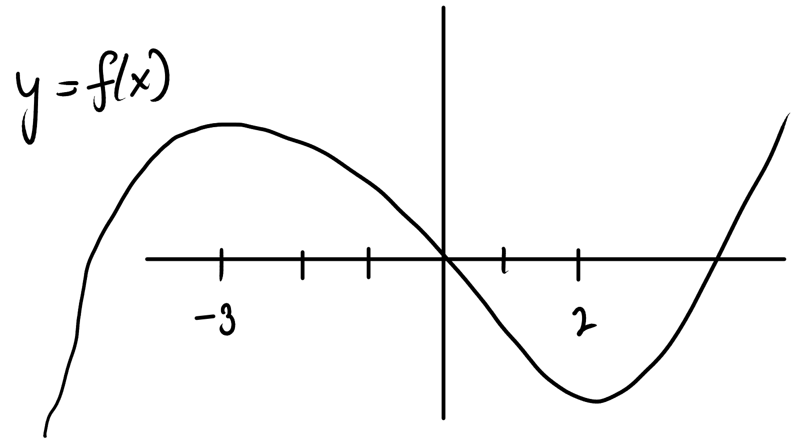
$f'(x)$:



$6(16) + 6(-4) - 36 = 96 - 24 - 36 = 96 - 60 > 0$

increasing: $(-\infty, -3)$ and $(2, \infty)$
decreasing: $(-3, 2)$

There is a local maximum at $x = -3$
there is a local minimum at $x = 2$



<u>Second derivative $f''(x)$</u> = rate of change of $f'(x)$

Assume $f'(c) = 0$

$f(c)$ is a local min $\Leftarrow$ f' goes from negative to positive

$\Nearrow$

f' is increasing $\Longleftarrow$ $f''(c) > 0$

$f''(c) < 0 \Rightarrow f'$ is decreasing

$\Rightarrow f'$ goes positive to negative $\Rightarrow f(c)$ is local max

Illustrate using previous problem

$f(x) = 2x^3 + 3x^2 - 36x$ $\qquad c = -3 : f''(-3) = -30$

$f'(x) = 6x^2 + 6x - 36$

$\qquad c = -3$ is local max

$f''(x) = 12x + 6$

$\qquad c = 2 : f''(2) = 30$

$\qquad c = 2$ is local min

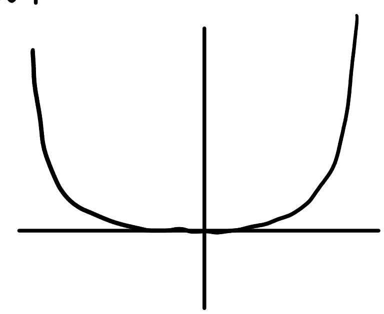
$f''(c) = 0 \implies$ inconclusive

$f(x) = x^4 \qquad f'(x) = 4x^3 \qquad f''(x) = 12x^2$

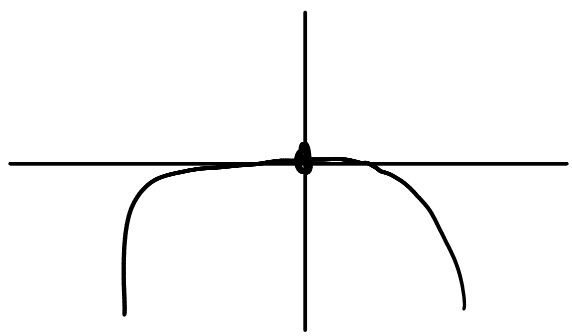
$\Downarrow$

$c = 0$ is critical point

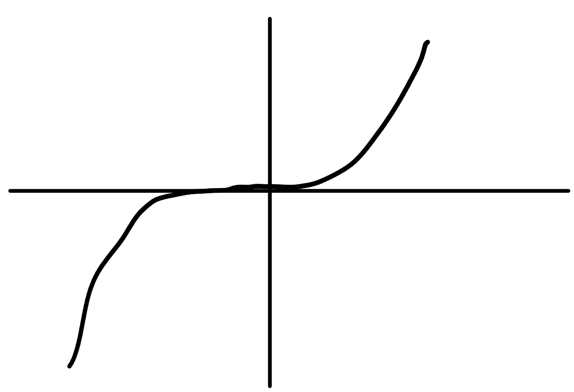
$f''(0) = 0$



$f(x) = -x^4 \qquad f'(0) = 0 \qquad f''(0) = 0$

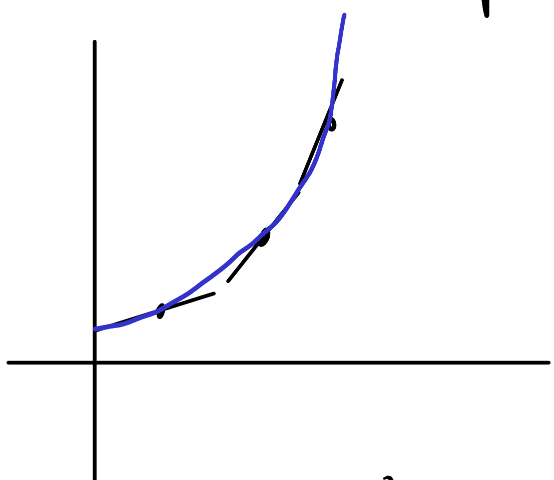


$f(x) = x^3 \qquad f'(0) = 0 \qquad f''(0) = 0$



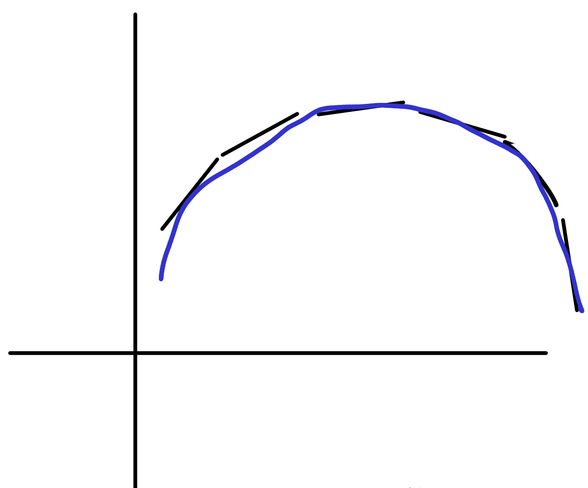
What does $f''(x)$ mean? concavity/convexity

$f''(x) > 0$ slopes increase



concave up

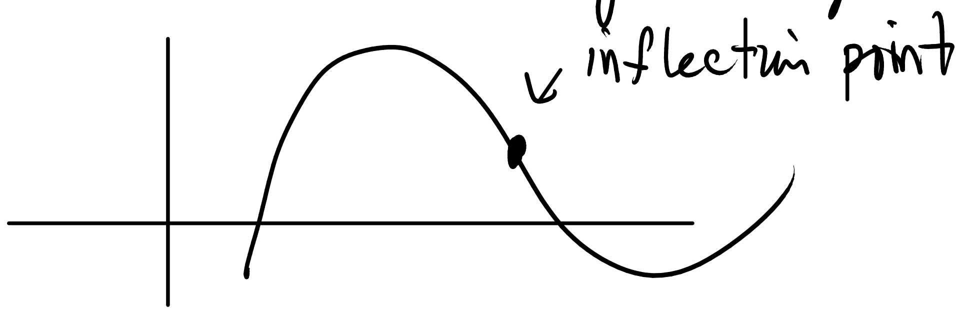
$f''(x) < 0$ slopes decrease

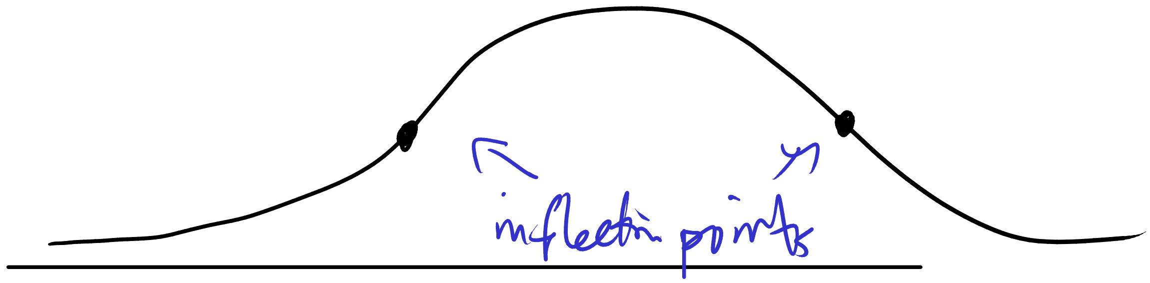


concave down

Points where $f''(x) = 0$ are called inflection points.

Points where concavity changes




inflection points