Maximum and Minimum Values.

$f(x)$ function on a specified Domain D

usually $D = [a,b] = \{a \le x \le b\}$

Let c be a point in the domain D

<u>Definition</u>: $f(c)$ is

1) an <u>absolute maximum</u> of f on D if

$f(c) \ge f(x)$ for all x in D

2) an <u>absolute minimum</u> of f on D if

$f(c) \le f(x)$ for all x in D

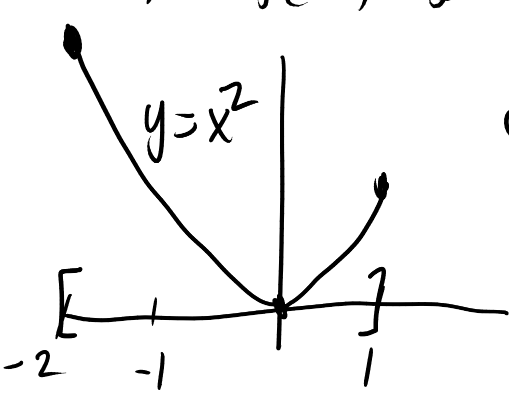
3) a <u>local maximum</u> if

$f(c) \ge f(x)$ for all x in D near c

4) a <u>local minimum</u> if

$f(c) \le f(x)$ for all x in D near c.

<u>Ex</u>



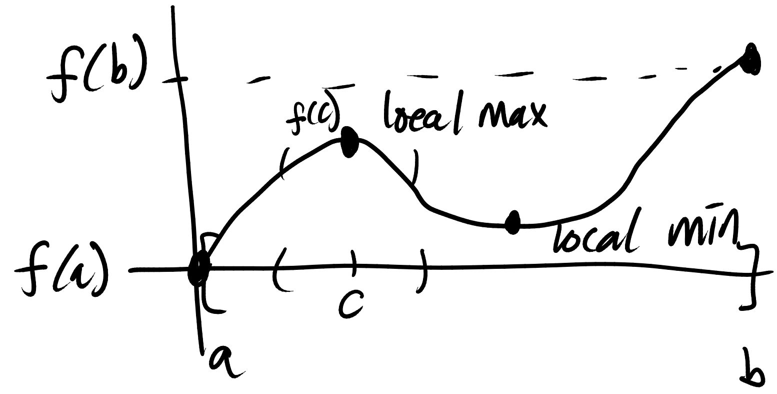
domain $D = [-2, 1]$

$c = -2$ is where the absolute maximum occurs

$f(-2) = 4$ is abs. max.

$c=0$ is where absolute minimum occurs
$f(0)=0$ is the absolute minimum.

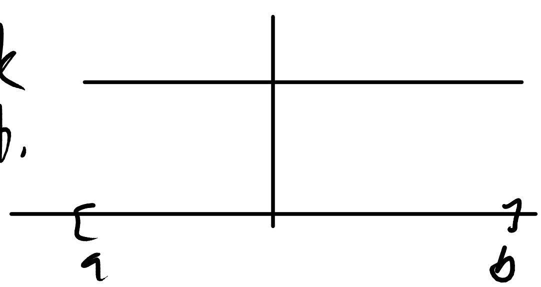
<u>Ex</u>



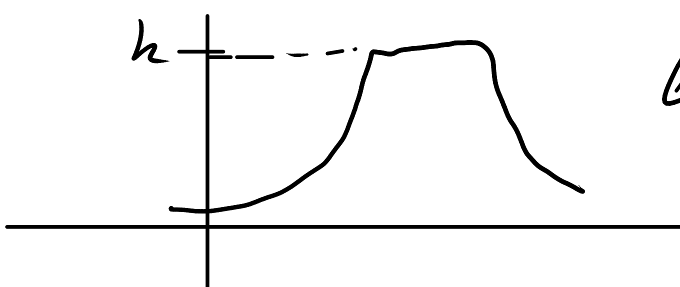
$f(a)$ = absolute min $\quad f(b)$ = absolute max

<u>Ex</u>

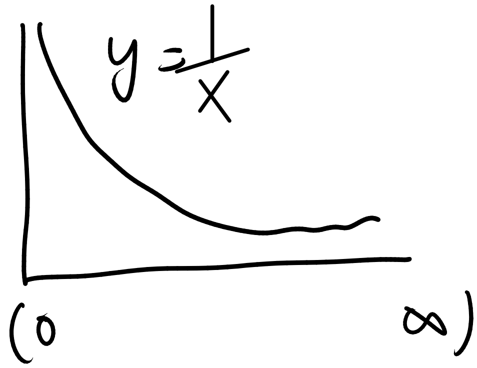
$f(x)=k$
constant.



max $=k$ min $=k$
maximum is achieved at many points.



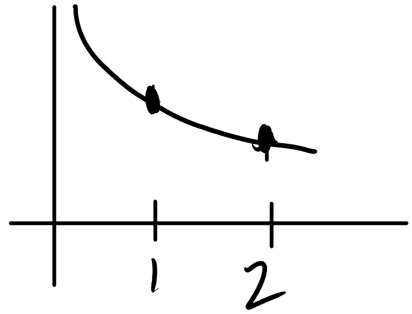
max/min
$f(x)=\frac{1}{x}$
$D=(0,\infty)$
has no maximum



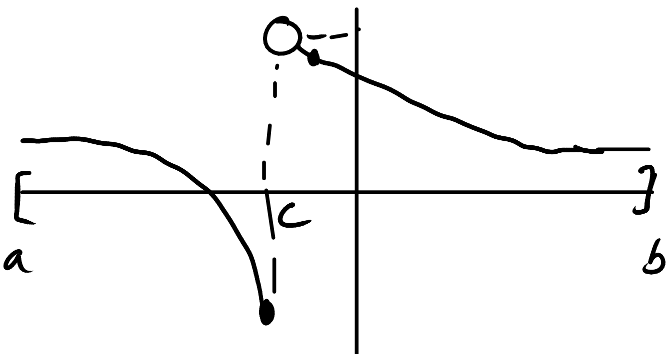
also has no minimum

$f(x) = \frac{1}{x}$

$D = [1,2]$

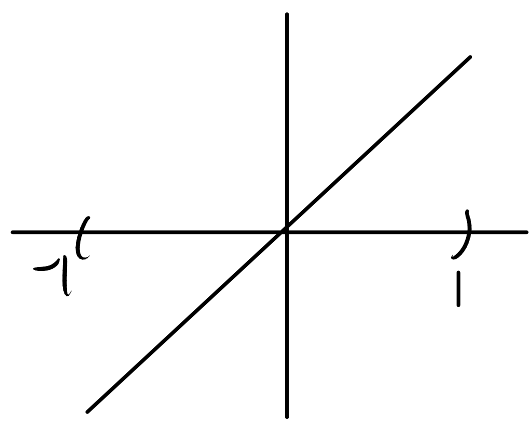


Ex



$f(c)$ = absolute minimum.
there is no absolute maximum

Ex
$f(x) = x$
$D = (-1,1)$

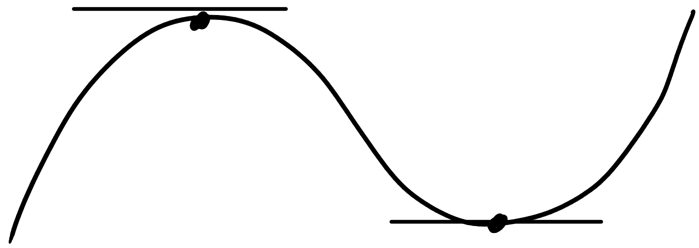


has no maximum
or minimum
on the domain
$D = (-1,1)$
excluding end points

Extreme value theorem
IF f is continuous on a closed interval $[a,b]$
THEN f attains an absolute maximum value at some point c in $[a,b]$

ALSO f attains an absolute minimum value at some point d in $[a,b]$

Q: how to find max/min?



observe $f'(c)=0$
if c is local max or min

Fermat's theorem: If f has a local max or min at c, and $f'(c)$ exists (that is, f is differentiable at c)

Then $f'(c)=0$

Proof Suppose $f(c)$ is a local max

$\Rightarrow$ $f(c) \geq f(c+h)$

$f(c+h)-f(c) \leq 0$

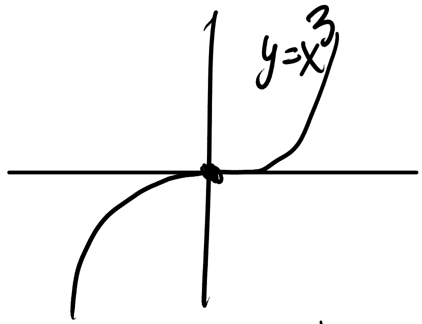
$$\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$$

$$\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

$f'(c)$ exists $\Rightarrow$ $0 \leq \lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0^+} (\text{"}) \leq 0$

$$\Rightarrow f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = 0$$

Consider

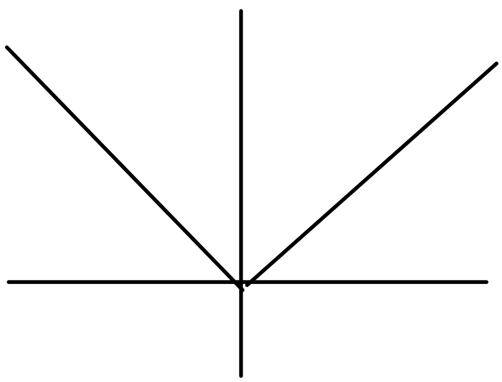


$f(x)=x^3$

$f'(x)=3x^2$

$f'(0)=0$

The converse of the theorem is not true.



$f(x)=|x|$

local min ? $=0$
at $x=0$

<u>Critical number</u> c is any number such that
either $f'(c)=0$ (and it exists)
or $f'(c)$ does not exist.

Also called "critical point"

Critical points <u>may</u> be local max or min

To find absolute maximum or minimum

1) find critical numbers and the the values of f at these numbers.

2) find values at endpoints

3) compare values, take biggest or smallest one.

Find absolute max and min:

<u>Ex</u> $f(x) = 12 + 4x - x^2$ on $[0, 5]$

$f'(x) = 4 - 2x$

Critical numbers $\quad f'(x) = 0 = 4 - 2x$

$x = 2 \qquad\qquad 4 = 2x$

$2 = x$

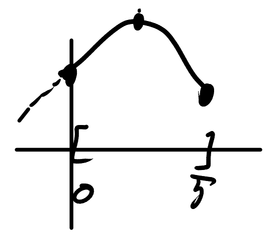
$f(2) = 12 + 8 - 4 = 16$

End points

$f(0) = 12 \qquad f(5) = 12 + 20 - 25 = 7$

Absolute max $= 16$ at $x = 2$

Absolute min $= 7$ at $x = 5$



$f(x) = x e^{-x^2/8} \qquad [-1, 4]$

$f'(x) = e^{-x^2/8} + x e^{-x^2/8}\left(-\frac{2x}{8}\right)$

$= e^{-x^2/8}\left(1 + x\left(\frac{-2x}{8}\right)\right) = e^{-x^2/8}\left(1 - \frac{x^2}{4}\right)$

$f'(x) = 0 = e^{-x^2/8}\left(1 - \frac{x^2}{4}\right)$

$e^{-x^2/8} = 0 \qquad$ OR $\qquad \left(1 - \frac{x^2}{4}\right) = 0$

No solution

$e^x > 0 \qquad\qquad x = \pm 2$

we don't care about $x=-2$ because the domain of interest is $[-1,4]$

$x=2$ $\quad f(2) = 2e^{-(2)^2/8} = 2e^{-1/2}$

End points $\quad f(-1) = (-1)e^{-(-1)^2/8} = -e^{-1/8}$

$f(4) = 4e^{-(4)^2/8} = 4e^{-2}$

Possible max/min : $2e^{-1/2}, -e^{-1/8}, 4e^{-2}$

Minimum is $-e^{-1/8}$ because it's the only one that is negative.

$\frac{2}{\sqrt{e}} \quad \frac{4}{e^2} \quad e \approx 2.71828\ldots \quad e^2 > 4$

$\frac{4}{e^2} < 1$

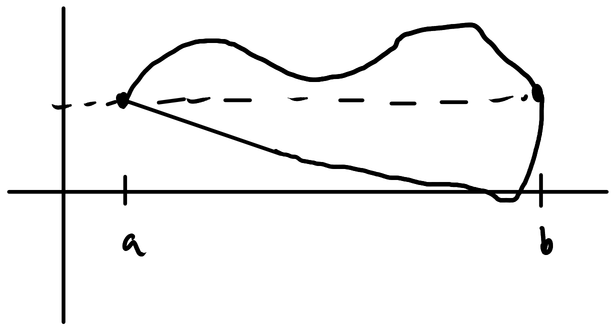
$$4e^{-2} \overset{<}{\text{vs}} 2e^{-1/2}$$
$$2e^{-2} \text{ vs } e^{-1/2}$$
$$2 \text{ vs } e^{-1/2} \cdot e^2 = e^{3/2}$$

$e^{3/2} > e^1 = e = 2.71828 > 2$

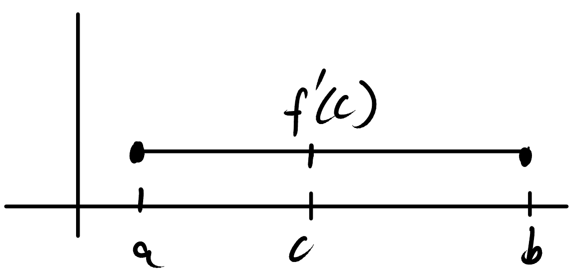
$4e^{-2} < 2e^{-1/2}$

Absolute maximum $= 2e^{-1/2}$ at $x=2$

Rolle's theorem investigate condition $f'(c)=0$



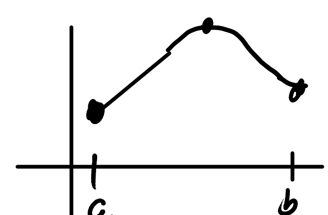
<u>Rolle's theorem</u> IF f is continuous on $[a,b]$
and f is differentiable on (a,b)
and $f(a)=f(b)$
THEN there is some c in (a,b) such that
$f'(c)=0$



Case where f is constant.

Suppose not constant

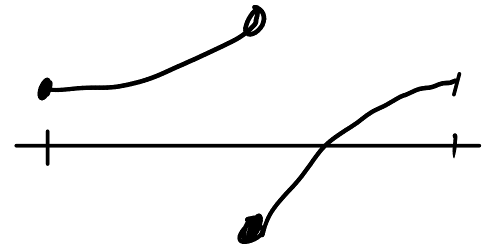
$f(x) > f(a)$ for some x in (a,b)
Extreme value theorem $\Rightarrow$
exists maximum $f(c)$, not at endpoints



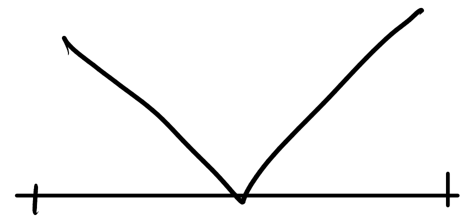
f is differentiable so Fermat's theorem implies
$f'(c)=0$.

$f(x) < f(a)$ for same x in (a,b)
exists minimum $f(c)$, not at endpoints
$f'(c)=0$ by Fermat's theorem.

Need continuous



Need differentiable



Need $f(a) = f(b)$