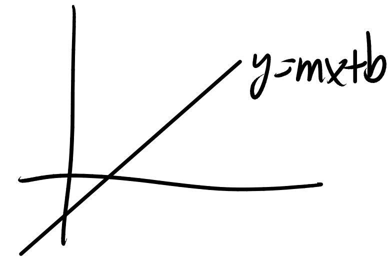
Linear approximation $\quad y = f(x)$

Linear function: $\quad y = mx + b$

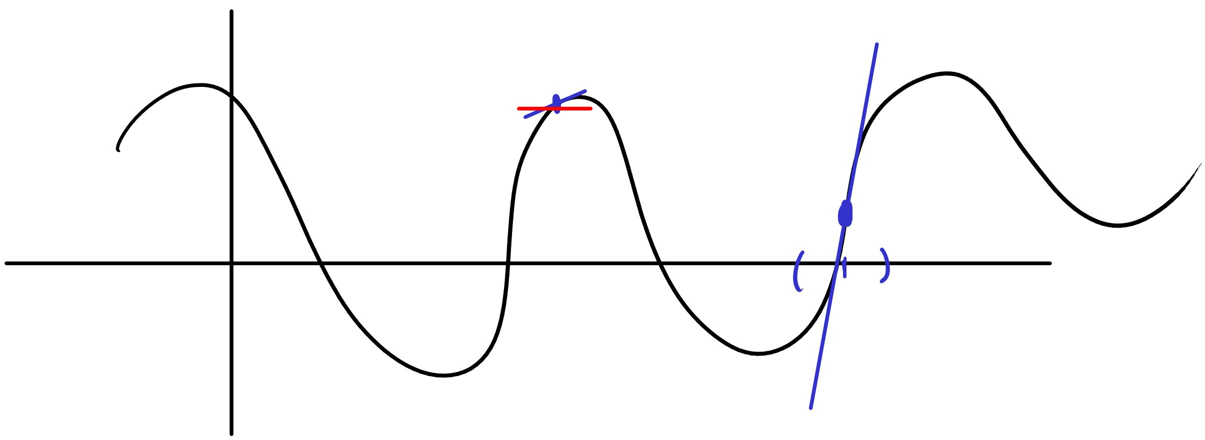
x appears to first power only.



What linear function best approximates $f(x)$?

$\Updownarrow$

(least error)

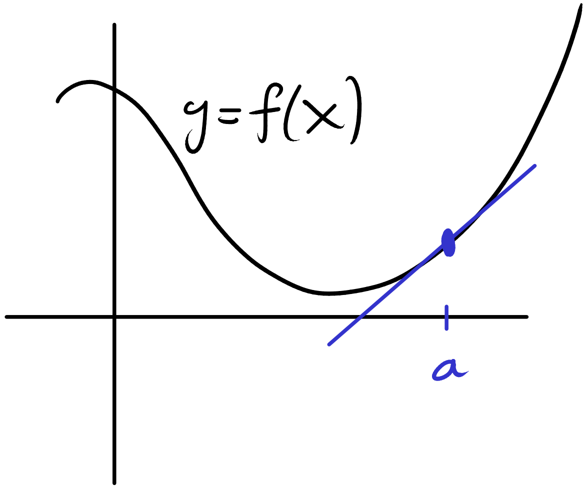


Want the approximation to be good in some interval, or near some point

Get a different linear approximation at each point.

Which line fits best at a given point?

Answer: Tangent line at the given point.



slope $f'(a)$

point $(a, f(a))$

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$

This is the linear approximation (tangent line approx) to $f(x)$ at $x=a$

The function $L(x) = f(a) + f'(a)(x-a)$ is the linearization of f at a.

$$L_{f,a}(x) = f(a) + f'(a)(x-a)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(x) \approx L(x)$$

<u>Ex:</u> $f(x) = \sqrt{x+3}$ linear approx at $a=1$ use it to find $\sqrt{3.98}$

Linear approx $f(a) + f'(a)(x-a)$

$$f(a) = f(1) = \sqrt{1+3} = \sqrt{4} = 2$$

$$f'(x) = \frac{d}{dx}\left[(x+3)^{1/2}\right] = \frac{1}{2}(x+3)^{-1/2} \cdot 1$$

$$f'(a) = f'(1) = \frac{1}{2}(1+3)^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{4}} = \frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x-1)$$ This is the linearization (linear approximation)

$$\sqrt{x+3} \approx 2 + \frac{1}{4}(x-1)$$ for x close to 1

$\sqrt{3.98}$ corresponds to $x = 0.98$
close to 1 ✓ use approximation

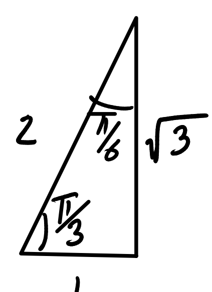
$$\sqrt{3.98} \approx 2 + \frac{1}{4}(.98-1) = 2 + \frac{1}{4}(-.02)$$

$$\left[\ \frac{1}{4}(.02) = \frac{1}{4}\,\frac{2}{100} = \frac{1}{200} = \frac{5}{1000} = .005\ \right]$$

$$\sqrt{3.98} \approx 2 - .005 = 1.995$$

Mathematica 1.99499

$f(x) = \sin x$ $\quad a = \pi/6$



$f(a) = \sin\frac{\pi}{6} = \frac{1}{2}$

$f'(x) = \cos x \qquad f'(a) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$$f(x) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$$ valid for x near $\pi/6$

Why is tangent line best?

$$\text{Error} = f(x) - L(x) = f(x) - \left(f(a) + f'(a)(x-a)\right)$$

$$= f(x) - f(a) - f'(a)(x-a)$$

Error approaches 0 as x approaches a.
But this is true for any slope!!

More is true

$$\frac{\text{Error}}{x-a} = \frac{f(x)-f(a)}{x-a} - f'(a)$$

$$\lim_{x\to a} \frac{\text{Error}}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - \lim_{x\to a} f'(a)$$

$$= f'(a) - f'(a) = 0 \quad !!!$$

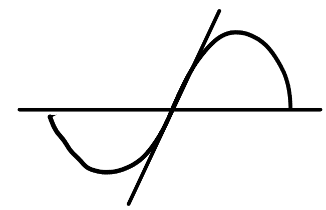
Error goes to zero faster than x−a goes to zero (as x goes to a)

This property is only true for the Linearization

Common approximations

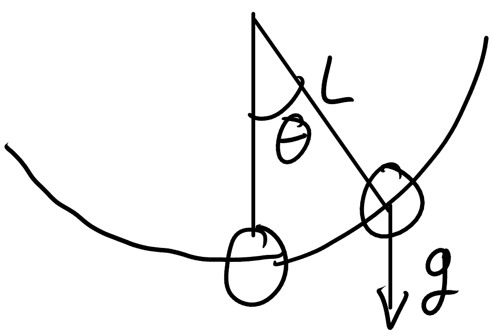
$f(x) = \sin x$ at $a = 0$ $\qquad \sin 0 = 0$

$f'(x) = \cos x$ $\qquad \cos 0 = 1$



$$L(x) = 0 + 1(x - 0) = x$$

Slogan: $\sin x \approx x$ if x is close to 0.



$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

suppose small motion

θ close to 0.

$\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

$$\theta = A\sin\left(\sqrt{\frac{g}{L}}\,t + \phi\right)$$

$f(x) = (1+x)^r$ $\qquad a = 0$ $\qquad f(0) = 1^r = 1$

$f'(x) = r(1+x)^{r-1}$ $\qquad f'(0) = r(1)^{r-1} = r$

$(1+x)^r \approx 1 + rx$ for x close to 0.

$\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for x close to 0

$\frac{1}{1+x} \approx 1 - x$ for x close to 0

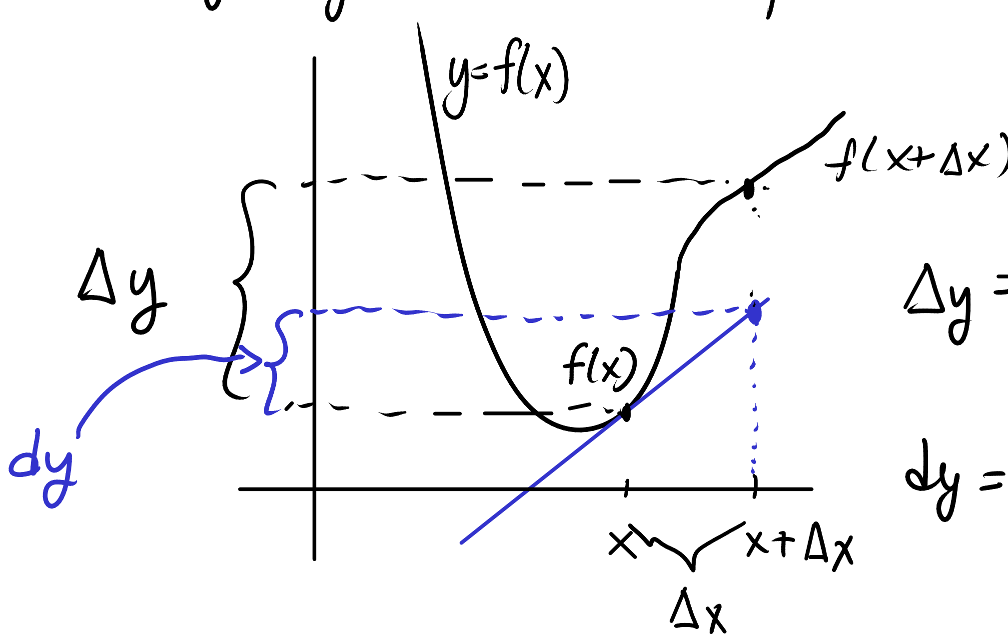
$$\frac{1}{1-x} = \frac{1}{1+(-x)} \approx 1-(-x) = 1+x$$

$$e^x \approx 1 + 1x = 1+x$$

Differentials: a different notation for linear approximations

$$y = f(x) \qquad \frac{dy}{dx} = f'(x) \implies dy = f'(x)dx$$

Think of dy and dx as separate variables.



Δy = change in function

dy = change in the linear approximation.

$$dx = \Delta x = \text{change in } x$$

$dx = \Delta x$

$$\Delta y = f(x+\Delta x) - f(x)$$
$$dy = f'(x)\Delta x = f'(x)dx$$

Linear approximation says $\Delta y \approx dy$

$$f(x+\Delta x) - f(x) \approx f'(x)\Delta x$$
$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

translate
$a \leftrightarrow x$
$x-a \leftrightarrow \Delta x$

Compare Δy and dy for $y = \underbrace{2x - x^2}_{f(x)}$ $x=2$, $\Delta x = -0.4$

$$\Delta y = f(x+\Delta x) - f(x) = f(1.6) - f(2)$$
$$= 2(1.6) - (1.6)^2 - 2\cdot 2 + 2^2$$
$$= .64$$

$$dy = f'(x)\,dx$$

$f'(x) = 2 - 2x$
at $x=2$
$f'(2) = -2$

$$dy = (-2)(-0.4)$$
$$= .8$$

Estimate $(1.999)^4$ $\qquad y = x^4$

center value

$x = 2$ $\qquad dy = 4x^3\,dx$

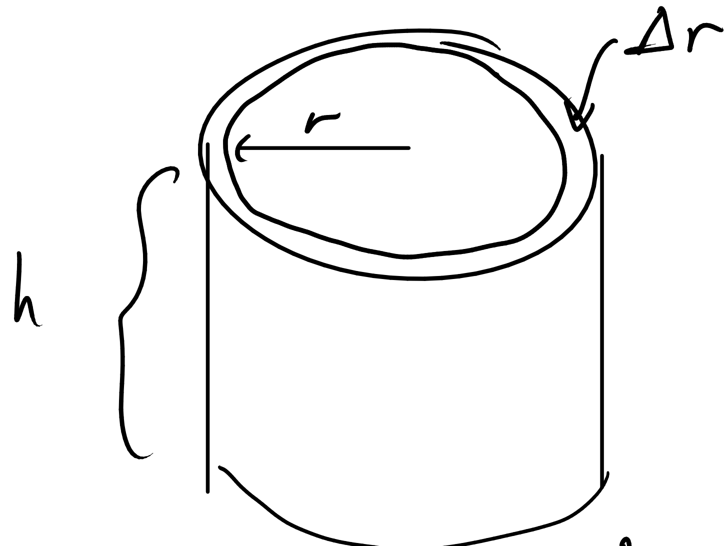
$\Delta x = -.001 \qquad x + \Delta x = 1.999$

$$dy = 4(2)^3(-.001) = -0.032$$

$$\Delta y = (x+\Delta x)^4 - (x)^4 = (1.999)^4 - 16 \approx dy = -.032$$

$$(1.999)^4 \approx 16 - .032 = 15.968$$

Approximate volume of a thin cylindrical shell inner radius r, height h, thickness Δr



$V = \pi r^2 h$

$\dfrac{dV}{dr} = 2\pi r h$

Volume of shell $= \pi (r+\Delta r)^2 h - \pi r^2 h$

$= \Delta V$ as r changes by Δr

$$\Delta V \approx \frac{dV}{dr}\Delta r = 2\pi r h \cdot \Delta r$$