Implicit differentiation.

Let's use it to find $\frac{d}{dx}\left(\cos^{-1} x\right)$

$$y = \cos^{-1} x$$

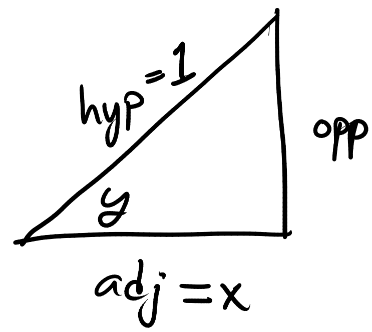
$$\cos y = x$$

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x) = 1$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sin(\cos^{-1}(x))}$$

$$\frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}$$



$$\sin y = \frac{opp}{hyp}$$

$$opp^2 + x^2 = 1^2$$

$$opp = \sqrt{1-x^2}$$

$$\sin y = \sqrt{1-x^2}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$y = \sec^{-1} x$

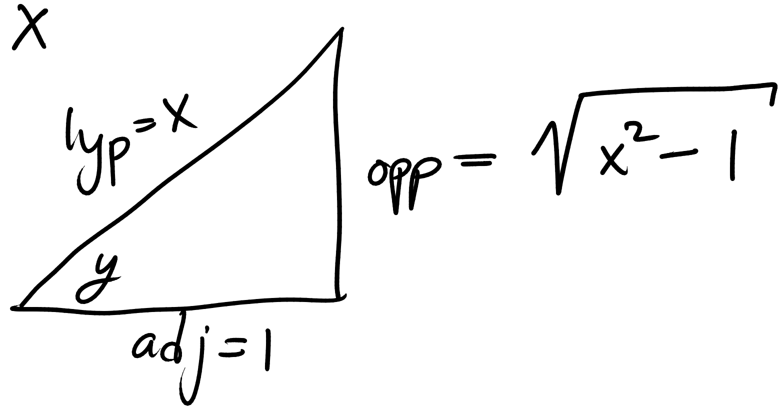
$\sec y = x$

$\frac{d}{dx}(\sec y) = \sec y \cdot \tan y \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y} \cdot \frac{1}{\tan y} = \cos y \cot y$

$= \cos y \cdot \frac{\cos y}{\sin y} = \frac{\cos^2 y}{\sin y}$

$\sec y = x$



$\sin y = \frac{\sqrt{x^2-1}}{x} \qquad \cos y = \frac{1}{x}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{\cos^2 y}{\sin y} = \left(\frac{1}{x}\right)^2 \frac{x}{\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ etc.

Logarithms.

$$\boxed{\frac{d}{dx}e^x = e^x}$$

$y = \ln x \qquad \frac{dy}{dx} = ?$

$e^y = x$

$$\frac{d}{dx}(e^y) = e^y \cdot \frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$y = \log_a x$

$a^y = x$

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(a^y) = \ln a \cdot a^y \cdot \frac{dy}{dx} = 1$$

$$\frac{d}{dx}(a^x) = (\ln a)\, a^x$$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(\ln a)x}\right)$$

$$= e^{\ln a \cdot x} \frac{d}{dx}(\ln a \cdot x)$$

$$= (\ln a) \cdot e^{\ln a \cdot x} = (\ln a) \cdot a^x$$

$$\frac{dy}{dx} = \frac{1}{\ln a \cdot a^y} = \frac{1}{(\ln a) \cdot x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \ln a}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x) = \frac{1}{\ln a} \cdot \frac{1}{x}$$

<u>Ex</u> $y = \ln(x^3+1)$

$u = x^3+1$

$\frac{du}{dx} = 3x^2$

$y = \ln u$

$\frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\,3x^2$$

$$= \frac{1}{x^3+1} \cdot (3x^2)$$

$$= \frac{3x^2}{x^3+1}$$

$y = \ln(x^3+1)$

$$\frac{dy}{dx} = \frac{1}{x^3+1}(3x^2)$$

$y = \ln(\sin x)$ $\qquad \frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$

$y = \ln(\sin^2 x)$ $\qquad \frac{dy}{dx} = \frac{1}{\sin^2 x}\frac{d}{dx}(\sin^2 x)$

$$= \frac{1}{\sin^{\cancel{2}} x}\, 2\cancel{\sin x}\cos x$$

$$= \frac{2\cos x}{\sin x} = 2\cot x$$

OR $\quad \sin^2 x = (\sin x)^2$

$$\ln(a^2) = 2\ln a \qquad \ln(a^b) = b\ln a$$

$$\ln(\sin^2 x) = \ln\left((\sin x)^2\right) = 2\ln(\sin x)$$

$$\frac{d}{dx}\left(\ln(\sin^2 x)\right) = 2\frac{d}{dx}\ln(\sin x) = 2\cot x$$

$$y = \ln(xe^x) = \ln x + \ln e^x$$

$$= \ln x + x$$

$$\frac{dy}{dx} = \frac{1}{x} + 1$$

$$y = \ln\left[\frac{(2x+1)^5}{\sqrt{x^2-1}}\right]$$

$$\frac{dy}{dx} = \left(\frac{(2x+1)^5}{\sqrt{x^2-1}}\right)^{-1} \frac{d}{dx}\left(\frac{(2x+1)^5}{\sqrt{x^2-1}}\right)$$

$$= \left(\quad \right)^{-1} \frac{\sqrt{x^2-1}\; 5(2x+1)^4 2 - \frac{1}{2\sqrt{x^2-1}}(2x)(2x+1)^5}{x^2-1}$$

$$y = \ln\left[\frac{(2x+1)^5}{\sqrt{x^2-1}}\right] = \ln(2x+1)^5 - \ln\sqrt{x^2-1}$$

$$= 5\ln(2x+1) - \frac{1}{2}\ln(x^2-1)$$

$$\frac{dy}{dx} = 5\,\frac{1}{2x+1}\cdot 2 - \frac{1}{2}\,\frac{1}{x^2-1}(2x)$$

$$= \frac{10}{2x+1} - \frac{x}{x^2-1}$$

logarithmic differentiation (a trick)

$y = x^x$ (1) Take ln of both sides

$$\ln y = \ln x^x = x \cdot \ln x$$

(2) Take derivative of both sides.

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{d}{dx}(x \cdot \ln x) = 1 \cdot \ln x + x\frac{1}{x} = \ln(x) + 1$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + 1$$

$$\frac{dy}{dx} = y\left(\ln(x)+1\right) = x^x\left(\ln(x)+1\right)$$

$$y = x^x = e^{x \ln x}$$

Take derivative using chain rule and other rules.

$$y = (\cos x)^{\sqrt{x}}$$

Take $\ln$: $\ln y = \ln\left[(\cos x)^{\sqrt{x}}\right] = \sqrt{x}\,\ln(\cos x)$

Take derivative:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x}\,\ln(\cos x)\right]$$

$$= \frac{1}{2\sqrt{x}}\ln(\cos x) + \sqrt{x}\,\frac{1}{\cos x}(-\sin x)$$

$$= \frac{1}{2\sqrt{x}}\ln(\cos x) - \sqrt{x}\,\tan x$$

$$\frac{dy}{dx} = y\left(\quad\right) = (\cos x)^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln(\cos x) - \sqrt{x}\,\tan x\right)$$

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

$$\ln y = \ln\left[\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right]$$

$$= \ln\left(x^{3/4}\right) + \ln\left(\sqrt{x^2+1}\right) - \ln(3x+2)^5$$

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{1}{x^2+1}(2x) - 5\frac{1}{3x+2}3$$

$$= \frac{3}{4}\frac{1}{x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$$

$$\frac{dy}{dx} = y\left(\frac{3}{4}\frac{1}{x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4}\frac{1}{x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Applications of derivatives

$\frac{dy}{dx}$ = rate of change of y with respect to x

<u>physics</u>: <u>particle</u> moving in 1-dimension

position $s(t)$ t = time (seconds)

s = position coordinate (meters)

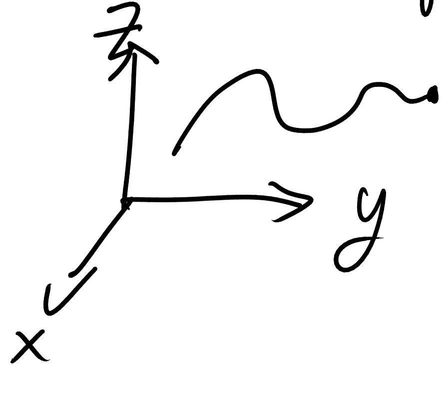
velocity $v(t) = s'(t) = \frac{ds}{dt}$ $\left(\frac{\text{meters}}{\text{second}}\right)$

acceleration $a(t) = v'(t) = \frac{dv}{dt}$ $\left(\frac{\text{meters}}{(\text{second})^2}\right)$

$= s''(t) = \frac{d^2s}{dt^2}$

Newton says $F = ma$

In actual physics position is determined by 3 functions of time



$x(t), y(t), z(t)$

$v_x(t), v_y(t), v_z(t)$

$a_x(t), a_y(t), a_z(t)$

Biology / Ecology N = # of animals in a population.

If population is very large, it can be described by a real number reasonably well.

Two species predators $N_{predator}$ (wolves)

prey N_{prey} (deer)

$$\frac{dN_{prey}}{dt} = N_{prey}(\alpha - \beta N_{predator})$$

$$\frac{dN_{predator}}{dt} = -N_{predator}(\gamma - \delta N_{prey})$$