Exam 15 quest problems 75%

2 free response 25%

Covers lectures 1–8

Solutions on website.

No books, notes or calculators

Chain rule, implicit differentiation and inverse functions

<u>Chain rule</u> compose function

$$y = f(x) = (2x+1)^2 = (\text{something})^2$$

where something $= 2x+1$

"something" $= u = 2x+1$ $\qquad y = f(x) = u^2$

u intermediate variable.

$$x \xrightarrow{u=2x+1} u \xrightarrow{y=u^2} y$$

Chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$$\frac{du}{dx} = \frac{d}{dx}[2x+1] = 2 \qquad \frac{dy}{du} = \frac{d}{du}[u^2] = 2u$$

$$\frac{dy}{dx} = \boxed{2u \cdot 2} = 2(2x+1)\cdot 2 = 4(2x+1) = 8x+4$$

$$f(x) = (2x+1)^2 \Rightarrow f'(x) = 8x+4$$

$$f(x) = (2x+1)^2 = 4x^2 + 4x + 1$$

$$f'(x) = 8x + 4$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$$

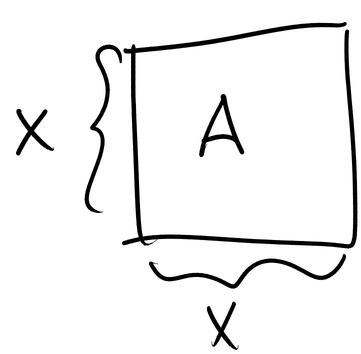
need to justify

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{dy}{du} \frac{du}{dx}$$

Mnemonic device $\quad \dfrac{dy}{dx} = \dfrac{dy}{\cancel{du}} \dfrac{\cancel{du}}{dx}$

Not a proof but very useful to remember. Kind of like dimensional analysis.

<u>Ex</u>



A = area

x = side length $\qquad A = x^2 \qquad \dfrac{dA}{dx} = 2x$

$x = 1 + e^t \qquad t$ = time.

$A = (1+e^t)^2 \qquad \dfrac{dx}{dt} = e^t$

$$\frac{dA}{dt} = \frac{dA}{dx} \frac{dx}{dt} = 2x(e^t) = 2(1+e^t)e^t$$

Another notation $\quad F(x) = f(g(x))$

[translate $\quad u = g(x)$
$\qquad\qquad\qquad y = f(u)$]

$$F'(x) = f'(\boxed{g(x)})\,g'(x)$$

f' is evaluated at $u = g(x)$

$F(x) = \sqrt{x^2+1} \qquad f(u) = \sqrt{u}$
$\qquad\qquad\qquad\qquad\quad g(x) = x^2+1$

$f'(u) = \dfrac{1}{2\sqrt{u}} \qquad g'(x) = 2x$

$$F'(x) = f'(g(x))\,g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}(x^2+1)^{1/2} = \frac{1}{2}(x^2+1)^{-1/2} \cdot \frac{d}{dx}[x^2+1]$$
$$= \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$$

$y = \sin(x^2) \qquad \dfrac{dy}{dx} = \cos(x^2)\dfrac{d}{dx}[x^2]$
$$= \cos(x^2) \cdot 2x$$

$u = x^2$

$y = \sin u \quad \dfrac{dy}{du} = \cos(u) = \cos(x^2)$

$\frac{du}{dx} = 2x \Rightarrow \frac{dy}{dx} = \cos(x^2) \cdot 2x$

$y = \sin^2 x = (\sin x)^2 = 2(\sin x)^1 \frac{d}{dx}[\sin x]$

$= 2 \sin x \cos x$

trig identity $= \sin 2x$

$y = \sin(\cos(x^2))$

$\frac{dy}{dx} = \cos(\cos(x^2)) \cdot \frac{d}{dx}[\cos(x^2)]$

$= \cos(\cos(x^2)) \cdot (-\sin(x^2)) \frac{d}{dx}[x^2]$

$= \cos(\cos(x^2)) \cdot (-\sin(x^2)) \cdot (2x)$

$y = \sin(\cos(x^2))$

$u = x^2$
$v = \cos u$
$y = \sin v$
$(x \to u \to v \to y)$

$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \cos v \cdot (-\sin u) \cdot 2x$

$= \cos(\cos u) \cdot (-\sin(x^2)) \cdot 2x$

$= \cos(\cos(x^2)) (-\sin(x^2)) 2x$

$$\boxed{\frac{d}{dx}\left[u^n\right] = n u^{n-1} \frac{du}{dx}}$$

$$\frac{d}{dx}\left[g(x)^n\right] = n\, g(x)^{n-1} g'(x)$$

Ex: $\frac{d}{dx}\left[(x^3-1)^{99}\right] = 99(x^3-1)^{98}(3x^2)$

Ex: $y = a^x$ $\quad \left[\frac{d}{dx}[e^x] = e^x\right]$

$$a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$$

$$\frac{d}{dx}\left[a^x\right] = \frac{d}{dx}\left[e^{(\ln a)x}\right] = e^u \cdot \frac{d}{dx}[(\ln a)x]$$

$$u = (\ln a)x$$

$$= e^{(\ln a)x} \cdot (\ln a) = a^x \cdot \ln a$$

$$\frac{d}{dx}\left[f(2x)\right] = f'(2x) \cdot 2$$

$$\frac{d}{dx} \sin(5x) = \cos(5x) \cdot 5 = 5\cos(5x)$$

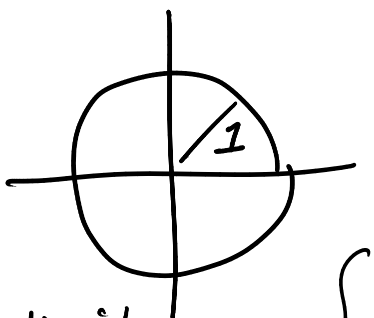
$$\frac{d}{dx} e^{7x} = e^{7x} \cdot 7 = 7e^{7x}$$

Implicit differentiation: useful, but confusing

"Implicit function":

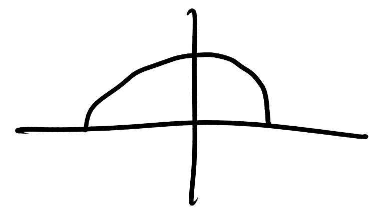
equation involving two variable x and y

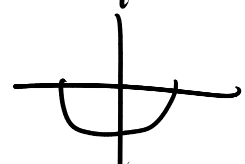
$x^2 + y^2 = 1$ If we solve for y we get y in terms of x.



$y = \pm\sqrt{1-x^2}$

explicit functions of x
$$\begin{cases} f(x) = \sqrt{1-x^2} \\ g(x) = -\sqrt{1-x^2} \end{cases}$$





These functions are implicitly defined by the equation
$$\left.\begin{aligned} x^2 + (f(x))^2 &= 1 \\ x^2 + (g(x))^2 &= 1 \end{aligned}\right\} \quad x^2 + y^2 = 1$$

The equation $x^2+y^2=1$ defines y as an implicit function of x

Example $x^2+y^2=-1$

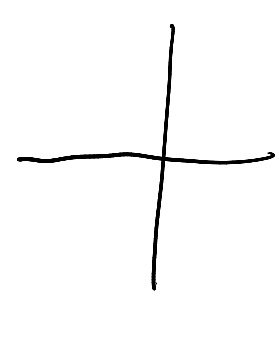
$y = \pm\sqrt{-1-x^2}$

$-1-x^2 > 0$

$-x^2 > 1$

$x^2 < -1$

impossible



In our class you're allowed to assume implicit function exists.

Nice thing: can compute derivative $\frac{dy}{dx}$ of the implicit function without solving explicitly for y.

$x^2+y^2=1$ find $\frac{dy}{dx}$ by implicit differentiation.

$$x^2+y^2=1$$

$$\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}[1] = 0$$

$$\frac{d}{dx}[x^2]+\frac{d}{dx}[y^2]$$

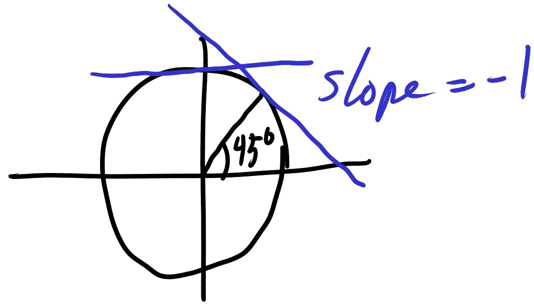
$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x \qquad \frac{dy}{dx} = \frac{-x}{y}$$

Find slope of tangent to $x^2+y^2=1$ at $\left(x=\frac{1}{\sqrt{2}}, y=\frac{1}{\sqrt{2}}\right)$

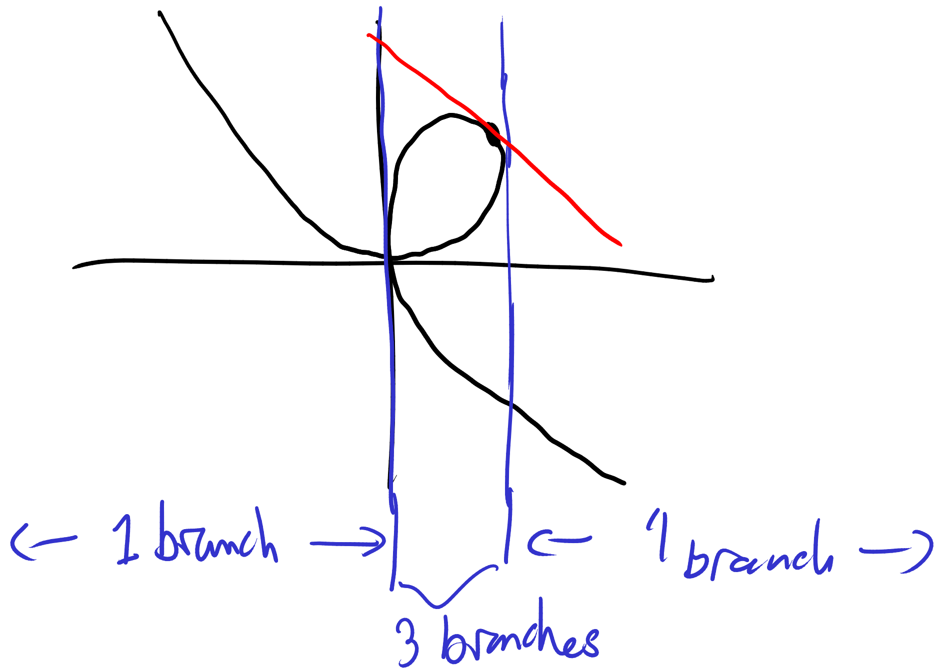
Check $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is actually on circle

$$\frac{dy}{dx} = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1$$



slope at $(x = 0, y = 1)$ $\frac{dy}{dx} = -\frac{0}{1} = 0$

$x^3 + y^3 = 6xy$



$$\frac{d}{dx}\left[x^3 + y^3\right] = \frac{d}{dx}\left[6xy\right]$$

$$\frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[y^3\right] = 6\left(\frac{d}{dx}[x] \cdot y + x \cdot \frac{d}{dx}[y]\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6\left(y + x\frac{dy}{dx}\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$\left(3y^2 - 6x\right)\frac{dy}{dx} = 6y - 3x^2$$

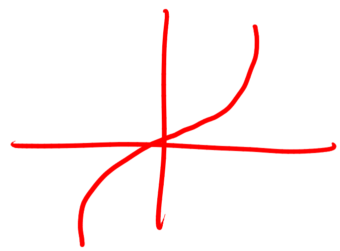
$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}$$

At $(x=3, y=3)$ $\quad \frac{dy}{dx} = \frac{18-27}{27-18} = -1$

Derivatives of inverse functions

$y = \sin^{-1} x$

$x = \sin y$



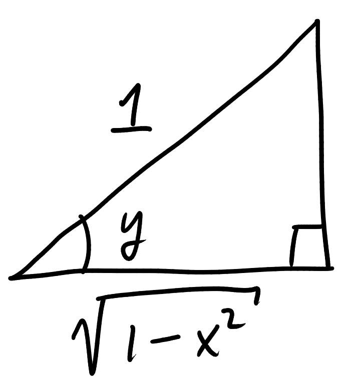
$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y] = (\cos y)\frac{dy}{dx}$$

$$\frac{d}{dx}[x] = 1$$

$$1 = \cos y \, \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}$$

Simplifying $\cos(\sin^{-1} x) = \cos y$



$\cos y = \sqrt{1-x^2}$

$\cos(\sin^{-1} x) = \sqrt{1-x^2}$