Computing derivatives

Notation: $y = f(x)$

Derivative

$$f'(x) \qquad y' \qquad \underbrace{\frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx} f(x)}_{\text{Leibniz notation}}$$

d is not a variable, just a symbol.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

set $h = \Delta x$ $\quad \Delta y = f(x+h) - f(x)$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

higher derivatives

$$f(x) \xrightarrow{\text{deriv.}} f'(x) \xrightarrow{\text{deriv}} f''(x)$$

$$\longrightarrow f'''(x) \longrightarrow f^{(4)}(x)$$

$$\longrightarrow \cdots \quad f^{(n)}(x)$$

Differentiation = taking the derivative.

$f(x) = c$ constant $\qquad f'(x) = 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

$f(x) = x \qquad f'(x) = 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

$f(x) = x^2 \qquad f'(x) = 2x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$(x+h)^2 = (x+h)(x+h) = x^2 + hx + xh + h^2 = x^2 + 2xh + h^2$$

$$\lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x+h)$$

$$= 2x$$

$$f(x) = x^n \qquad f'(x) = nx^{n-1}$$

Binomial Coefficient:

$$\binom{n}{k} = n \text{ choose } k = \frac{n!}{k!\,(n-k)!}$$

$$\binom{4}{2} = 4 \text{ choose } 2 = \frac{4!}{2!\,2!} = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot 2\cdot 1} = 6$$

$n=4 \quad k=2$

Binomial theorem

$$(x+h)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots$$

$$\cdots + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n$$

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1 \qquad \binom{n}{1} = \frac{n!}{1!\,(n-1)!} = n$$

Eg. $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

$$\boxed{f(x) = x^n} \qquad \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = f'(x)$$

$$= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots\right) - x^n}{h}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots}{h}$$

$$= \lim_{h\to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \cdots$$

$$= nx^{n-1} + 0 + 0 + 0$$

$$= nx^{n-1}$$

Note all other terms have positive power of h so they go to zero.

Eg. $f(x) = x^{1000}$ $\qquad f'(x) = 1000x^{999}$

$y = t^4$ $\qquad \frac{dy}{dt} = 4t^3$

Rules $\quad \frac{d}{dx}[c] = 0 \qquad \frac{d}{dx}[x^n] = nx^{n-1}$

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[4x^{99}+2x^3+7x^{50}+1]$$

$$= \frac{d}{dx}[4x^{99}] + \frac{d}{dx}[2x^3] + \frac{d}{dx}[7x^{50}] + \frac{d}{dx}[1]$$

$$= 4\frac{d}{dx}[x^{99}] + 2\frac{d}{dx}[x^3] + 7\frac{d}{dx}[x^{50}] + \frac{d}{dx}[1]$$

$$= 4\cdot 99\cdot x^{98} + 2\cdot 3\cdot x^2 + 7\cdot 50\cdot x^{49} + 0$$

<u>More generally</u>: $\frac{d}{dx}[x^r] = rx^{r-1}$

where r is any real number.

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-2} = \frac{-1}{x^2}$$

$$\frac{d}{dx}(x^{-2}) = -2x^{-3}$$

$$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(x^{\pi}) = \pi x^{\pi-1}$$

$$\frac{d}{dx}(\sqrt{x}^{11})$$

Exponential functions: $f(x) = a^x$ (not x^a)

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \qquad a^{x+h} = a^x a^h$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x (a^h - 1)}{h}$$

$$= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

doesn't depend on x. doesn't depend on h
only depends on a.

$$f(x) = a^x$$

$$f'(x) = (\text{something depending on } a) \cdot a^x$$

$$= f'(0)\, a^x$$

Derivative of exponential is proportional to itself

Chose a particular a that makes the proportionality an equality

e Euler's number base of natural logarithm.

$e = 2.718281828\ldots$

Theorem $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$

$f(x) = e^x$ then $f'(x) = e^x$

Problem $y = e^x$ find the point where the tangent line has slope 3.

$3 = y' = e^x$

$x = \ln 3$
$y = 3$

Product rule

$(fg)' \neq f'g'$

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}[g(x)] + \frac{d}{dx}[f(x)] \cdot g(x)$$

$u = f(x)$
$v = g(x)$

$(uv)' = u \cdot v' + u' \cdot v$

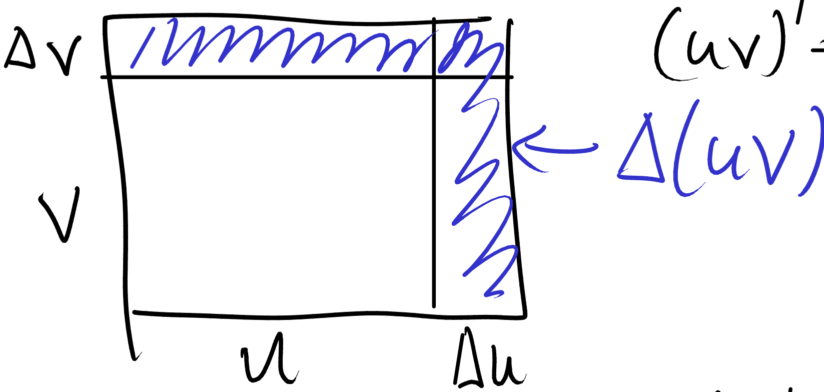
$\Delta x = h$

	x	versus	$x + \Delta x$
$f(x) = u$		$\longrightarrow$	$u + \Delta u = f(x+h)$
$g(x) = v$		$\longrightarrow$	$v + \Delta v = g(x+h)$

$\Delta u = f(x+h) - f(x)$
$\Delta v = g(x+h) - g(x)$

$uv \longrightarrow (u + \Delta u)(v + \Delta v)$

$$\Delta(uv) = (u+\Delta u)(v+\Delta v) - uv$$
$$= uv + v\Delta u + u\Delta v + \Delta u\Delta v - uv$$
$$= v\Delta u + u\Delta v + \Delta u\Delta v$$



$$(uv)' = \frac{d}{dx}(uv) = \lim_{\Delta x\to 0}\frac{\Delta(uv)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\left[v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right]$$

use fact $\displaystyle\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = u' = \frac{du}{dx}$

$$\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} = v' = \frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} + 0\cdot\frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$(uv)' = vu' + uv'$$

<u>Quotient rule:</u> $\displaystyle\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$$= \frac{1}{v}u' - \frac{u}{v^2}v'$$