Derivatives

$f(x)$ function
a point in domain

The derivative of f at a is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$ if this limit exists

OR

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$ if this limit exists

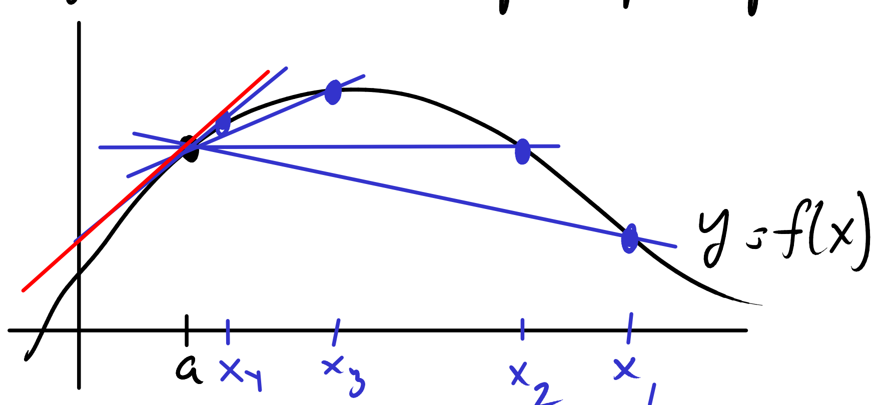
and the value of the limit is denoted $f'(a)$.

$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ equivalent to $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

via the substitution $h = x-a$
$x = a+h$

Geometric interpretation = slope of tangent line to graph

slope of tangent line = limit of slopes of the secant lines



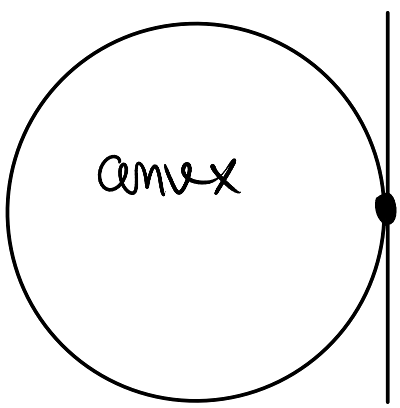
slope of secant line "between a and x"

line between $(a, f(a))$ $(x, f(x))$

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{f(x)-f(a)}{x-a}$$

Therefore slope of tangent line is

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$



Other interpretations of derivative

Physically y = position of a particle

t = time

$y=f(t)$ dependence of position on time

Then $f'(a)$ = instantaneous velocity of the particle at time $t=a$

$$\frac{\Delta y}{\Delta t} = \frac{f(t)-f(a)}{t-a} = \text{average velocity}$$

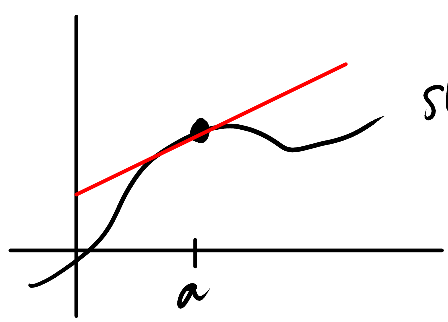
instantaneous velocity = limit of average velocities as $t \to a$.

more generally $f'(a)$ = instantaneous rate of change of $f(x)$ with respect to x when $x=a$.

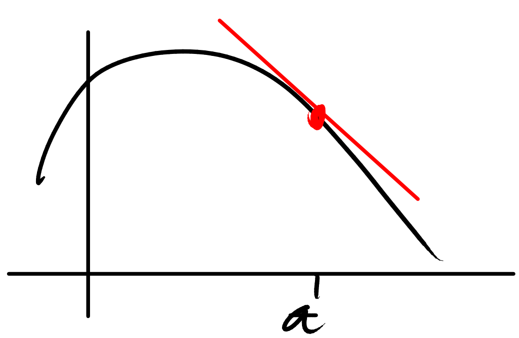
E.g. $T(t)$ = temperature of a hot object in a cool room.

$T'(t)$ = rate of change of temp with respect to time

Newton's law of cooling $T' = k(T_{room} - T)$



slope $= f'(a) > 0 \Rightarrow f(x)$ increasing



slope $f'(a) < 0 \Rightarrow f(x)$ decreasing.

velocity = rate of change of position coordinate with respect to time.

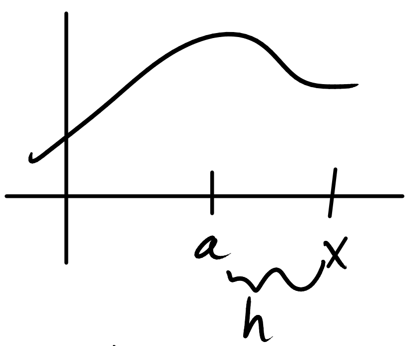
Negative velocity represents.

Decide $\longrightarrow$ right is the positive direction then negative velocity means moving left

If ↑ up is positive then negative velocity means moving down.

speed $= |\text{velocity}|$

E.g. $f(x) = \frac{1}{x}$ Find $f'(1)$ from the definition.
$a=1$



$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

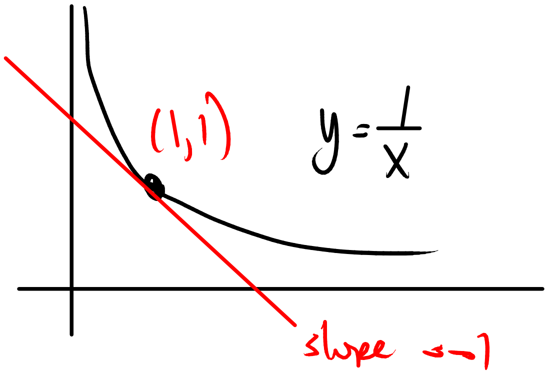
simplify $\frac{f(1+h)-f(1)}{h} = \frac{\frac{1}{1+h} - \frac{1}{1}}{h} = \frac{\frac{1}{1+h} - \frac{1+h}{1+h}}{h}$

$$= \frac{\left(\frac{1-(1+h)}{1+h}\right)}{h} = \frac{\left(\frac{1-1-h}{1+h}\right)}{h} = \frac{\left(\frac{-h}{1+h}\right)}{h}$$

$$= \frac{1}{h}\left(\frac{-h}{1+h}\right) = \frac{-1}{1+h}$$ if $h \neq 0$. and $h \neq -1$

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{-1}{1+h} = \frac{-1}{1+0} = -1$$

If $f(x) = \frac{1}{x}$ $f'(1) = -1$

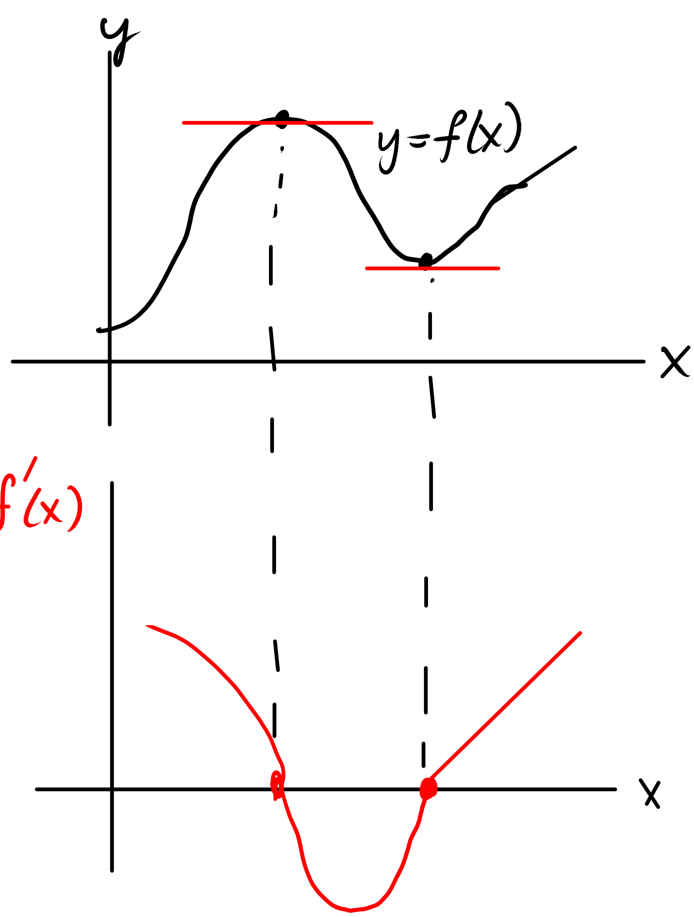


We can vary the point a where the derivative is computed call the variable x, get a function

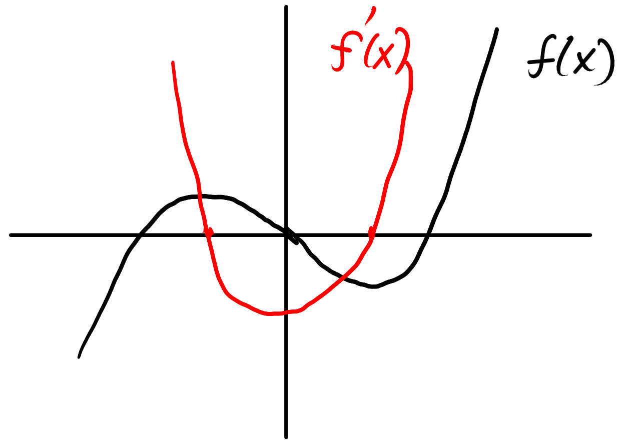
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

a function of (it may not be defined everywhere)

Eg.



$f(x) = x^3 - x \qquad f'(x) = 3x^2 - 1$



$$f(x) = \sqrt{x} \qquad f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

in this limit, x is treated as a constant.

$$\frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} = \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

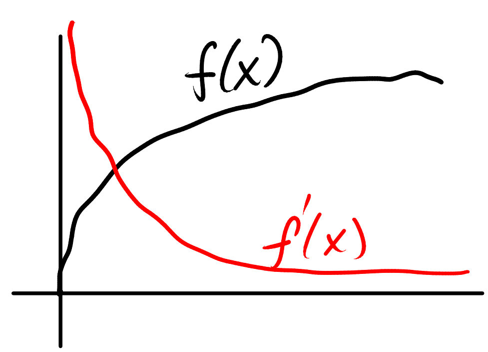
$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

works if $x > 0$

Does not exist at $x = 0$

$f(x) = \sqrt{x} \qquad f'(x) = \dfrac{1}{2\sqrt{x}}$



$f(x)$ is differentiable at a if $f'(a)$ exists.

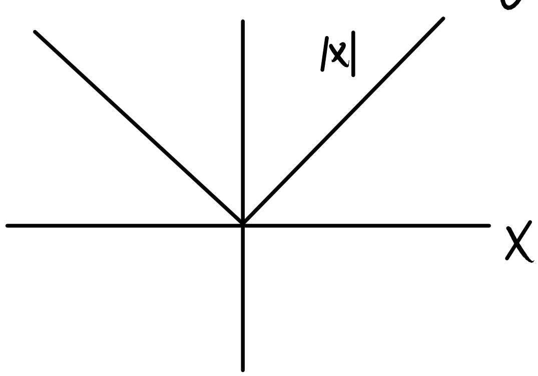
$f(x)$ is differentiable on (a,b) if its diff at every point in (a,b)

E.g. $f(x) = \sqrt{x}$ is differentiable on $(0,\infty)$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$f(x) = x^2 \qquad f'(x) = 2x \qquad$ differentiable on $(-\infty,\infty)$

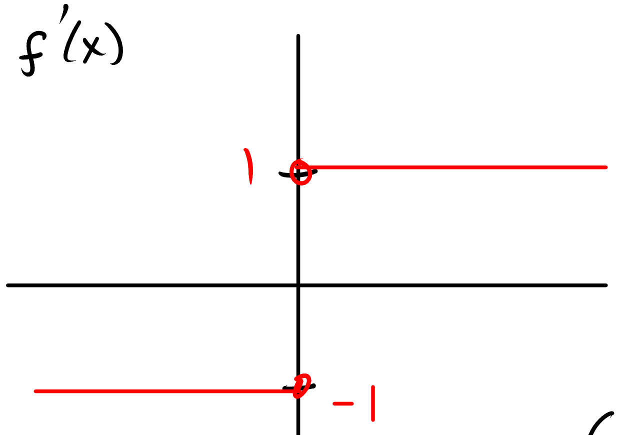
Failures of differentiability $\qquad f(x) = |x|$



Derivative at 0

$$\lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

This limit does not exist



That means $f(x) = |x|$ is not differentiable at $x=0$.

Call this a "sharp corner"

Theorem: if $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$

Proof: $\lim\limits_{x\to a}\left[f(x)-f(a)\right]=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}\cdot(x-a)$

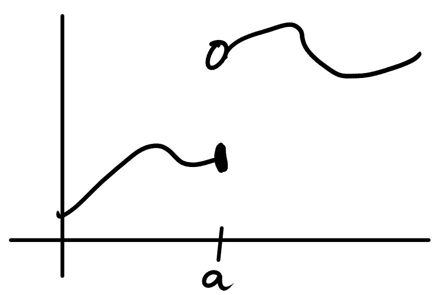
$=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}\cdot\lim\limits_{x\to a}(x-a)$ Because f diff at a.

$=f'(a)\cdot 0=0$

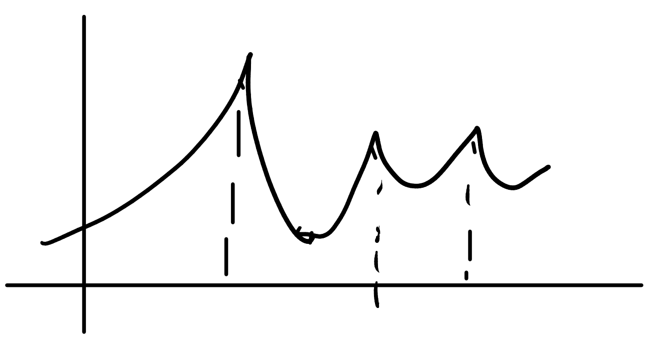
$\lim\limits_{x\to a}[f(x)-f(a)]=0$

$\lim\limits_{x\to a}f(x)=f(a)$

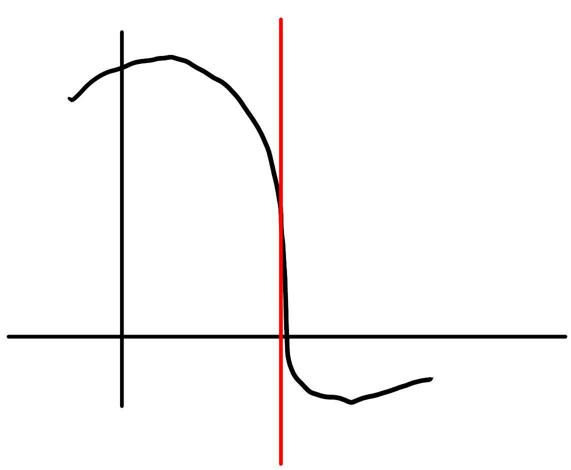
Failures



discontinuous at a $\Rightarrow$ not differentiable at a.



sharp corner / cusp not differentiable at those points.



vertical tangent $\Rightarrow$ slope not defined f is not differentiable

e.g. $f(x)=\sqrt[3]{x}$

Extreme oscillations