Continuity $f(x)$ point a

$f(x)$ is continuous at a if

(i) $f(a)$ exists

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

The limit exists and is equal to the value of the function

$f(x) = x$ Defined anywhere

does limit exist $\lim_{x \to a} x = a$ yes

$$\lim_{x\to a} x = \lim_{x \to a} f(x), \quad a = f(a)$$

$f(x) = \dfrac{x^2 - x - 2}{x-2}$ Not defined at $x=2$

If $x \neq 2$ $f(x)$ exists

Q: does $\lim_{x \to a} \dfrac{x^2-x-2}{x-2}$ exist if $a \neq 2$

x has a limit

x^2 has a limit

x^2-x-2 has a limit

$x-2$ has limit

as long the limit of $x-2$ is not zero

$\dfrac{x^2-x-2}{x-2}$ has a limit.

$\lim_{x\to a} x = a \quad \lim_{x\to a} x^2-x-2 = a^2-a-2$

$\lim_{x\to a} x-2 = a-2$

as long as $a-2 \neq 0$,

$\lim_{x\to a} \dfrac{x^2-x-2}{x-2} = \dfrac{a^2-a-2}{a-2} = f(a)$

$\overset{\shortparallel}{\lim_{x\to a} f(x)}$ as long as $a \neq 2$

$f(x) = \dfrac{x^2-x-2}{x-2}$ is continuous except at $a=2$.

for every $\varepsilon>0$, there is $\delta>0$ so that

$0<|x-a|<\delta$ implies $|x-a|<\varepsilon$

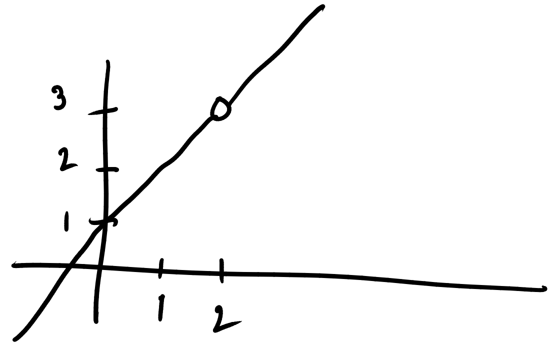
Answer take $\delta=\varepsilon$.

$\dfrac{x^2-x-2}{x-2} = \dfrac{(x+1)(x-2)}{x-2} = x+1$

valid for $x \neq 2$

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} x + 1 = 2 + 1 = 3$$

Looks something like



Consider

$$g(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

this function is continuous:

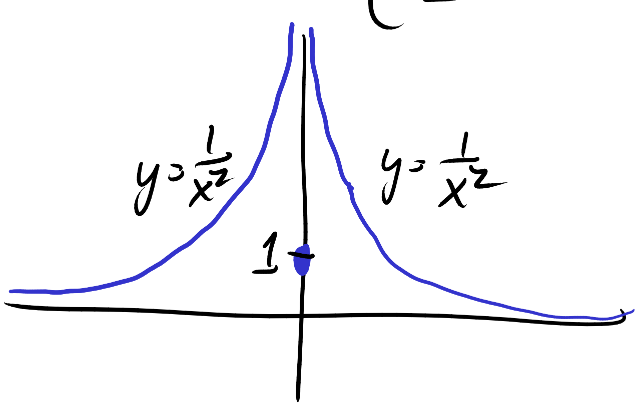
Reason $\lim_{x \to 2} g(x) = \lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} = 3 = g(2)$

So it is continuous at 2.

$$g(x) = x + 1 \quad \text{for every value of } x$$

Example

$$f(x) = \begin{cases} \dfrac{1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



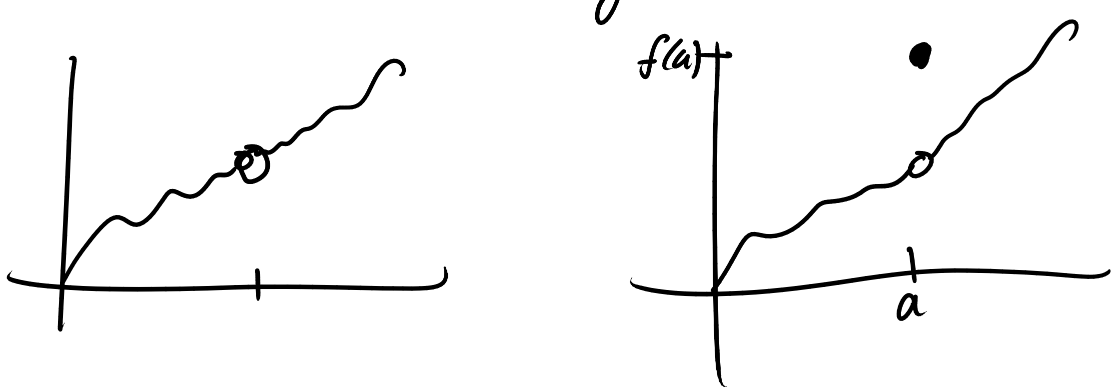
$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Not a number

$$f(0) = 1 \neq \infty = \lim_{x \to 0} f(x)$$

This function is discontinuous at 0 and can't be "fixed"

Removable discontinuity



$\lim_{x \to a} f(x)$ exists but $\neq f(a)$

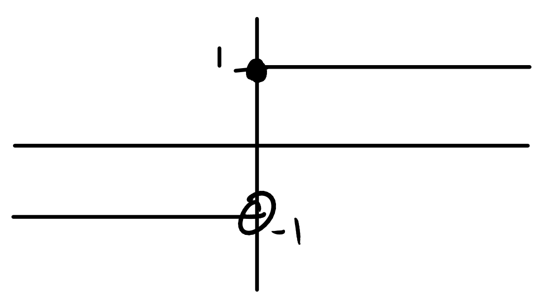
Can remove the discontinuity by redefining the function at a.

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ \lim_{x \to a} f(x) & \text{if } x = a \end{cases}$$

vertical asymptote $\iff$ Infinite discontinuity
Not removable.

jump discontinuity

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

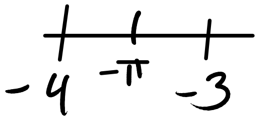


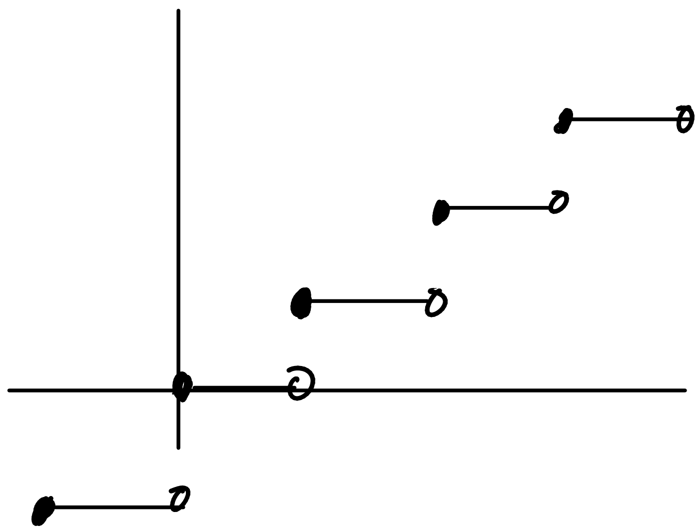
$f(x) = [[x]] =$ greatest integer $\leq x$

$f(0) = 0 \qquad f(1) = 1 \qquad f(2) = 2$

$f(\tfrac{1}{2}) = 0 \qquad f(\tfrac{3}{2}) = 1 \qquad f(\pi) = 3$

$f(-\pi) = -4$





This function is continuous at every point that is not an integer.

$\sin \frac{1}{x}$ has another type of discontinuity

One sided continuity

Continuous from the right means $\lim_{x \to a^+} f(x)$ exists & $f(a)$ exists &

$\lim_{x \to a^+} f(x) = f(a)$

Continuous from the left means

$\lim_{x \to a^-} f(x) = f(a)$ (and both sides exist)

$f(x) = [[x]]$ is continuous from the right at every point.

Not continuous from the left if a is an integer

- f is continuous on an interval $(a,b)=\{a<xb\}$

  if f is continuous at every point of this interval

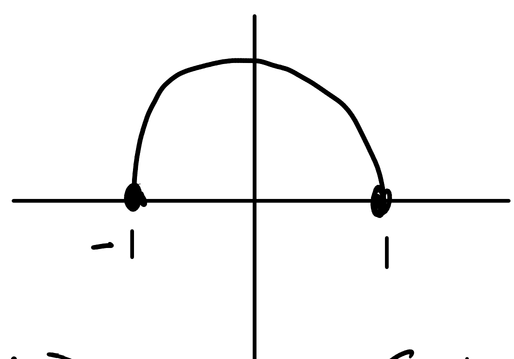
- f is continuous on closed interval $[a,b]=\{a\le x\le b\}$

  if f is continuous at every point of (a,b)

  and f is continuous from the right at a
  and f is continuous from the left at b.

$f(x)=\sqrt{1-x^2}$

Domain $= [-1,1]$



This function is continuous on $[-1,1]$

Suppose f and g are continuous at a.

Then the following functions are also continuous at a

$f+g$, $f-g$, cf, $f\cdot g$, $\frac{f}{g}$ as long as $g(a)\neq 0$.

Continuous

$x \quad x^2 \quad x^3$

$2x \qquad x^2+x^3+x^4 \qquad$ any polynomial

$3x$

$\dfrac{f(x)}{g(x)}$, where f and g are polynomials is continuous where ever denominator is not zero

a rational function is continuous on its domain.

Also continuous root functions $\sqrt[n]{\ }$ (on domain)

trig sin cos tan inverse trig $\sin^{-1}$ $\cos^{-1}$ $\tan^{-1}$

e^x continuous on $(-\infty, \infty)$

$\ln x$ continuous on $(0, \infty)$

Fact if g is continuous at a and

f is continuous at $g(a)$ then

$f(g(x)) = (f \circ g)(x)$ is continuous at a

For a continuous function (at a)

$$\lim_{x \to a} f(x) = f(a)$$

$$\lim_{x\to 5} \frac{e^{x^2}}{1+\ln(x^2+2)} = \frac{e^{5^2}}{1+\ln(5^2+2)}$$

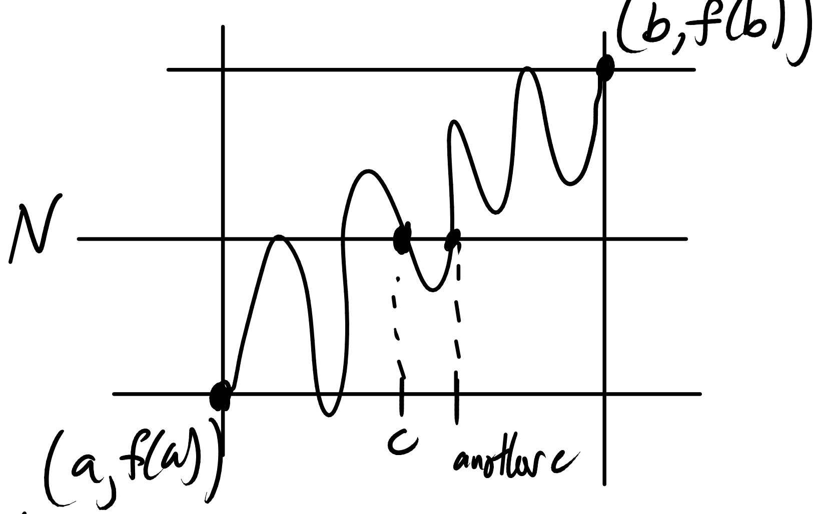
Because this is a continuous function.
Reason: it is built out of continuous functions using the operations $+, -, \cdot, \div, \circ$

Intermediate value theorem

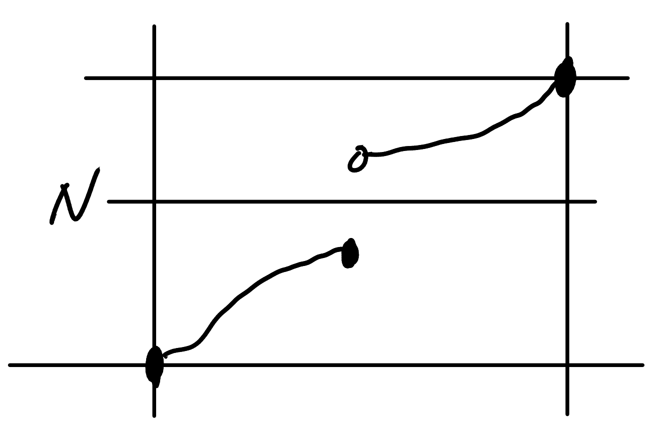
Suppose f is continuous on $[a,b]$

Let N be a number between $f(a)$ and $f(b)$

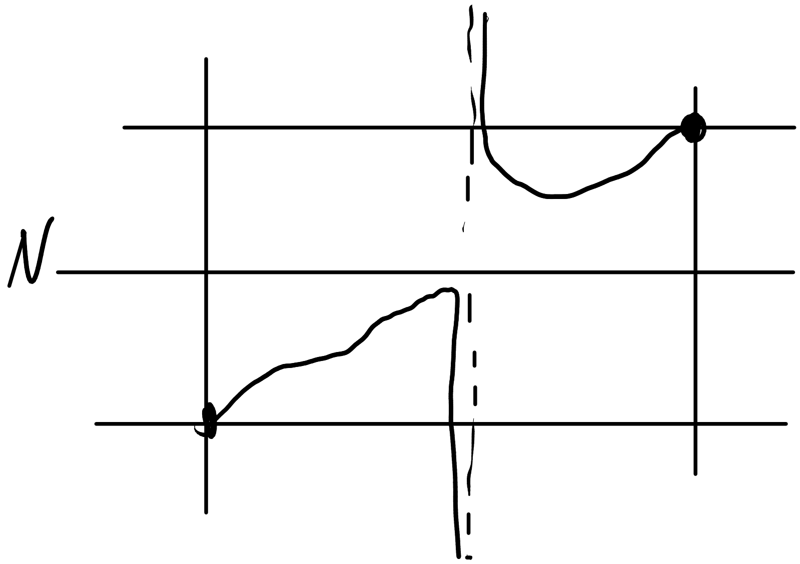
Then there is a c, $a \le c \le b$ where $f(c)=N$

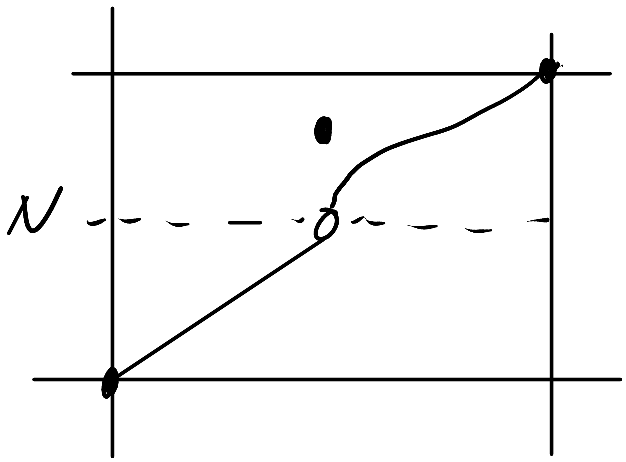


Non examples



Not continuous
$f(c)=N$
never happens.



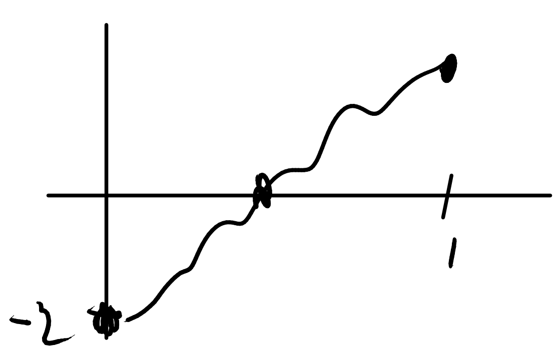


Prove that $e^x = 3-2x$ has a solution x in the interval $(0,1)$

$f(x) = e^x - 3 + 2x$ Trying to solve $f(x)=0$

$f(x)$ is continuous on $[0,1]$.

look $f(0) = 1 - 3 = -2 < 0$
$f(1) = e - 3 + 2 = e - 1 = 1.7828 > 0$



Intermediate value theorem implies there is a c in $(0,1)$ such that $f(c)=0$.