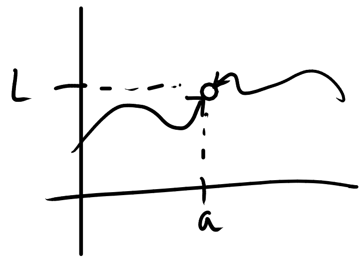
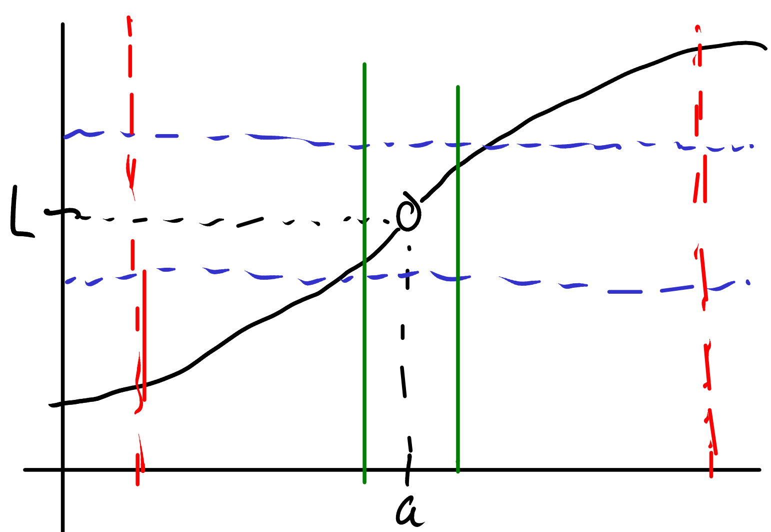
Limits

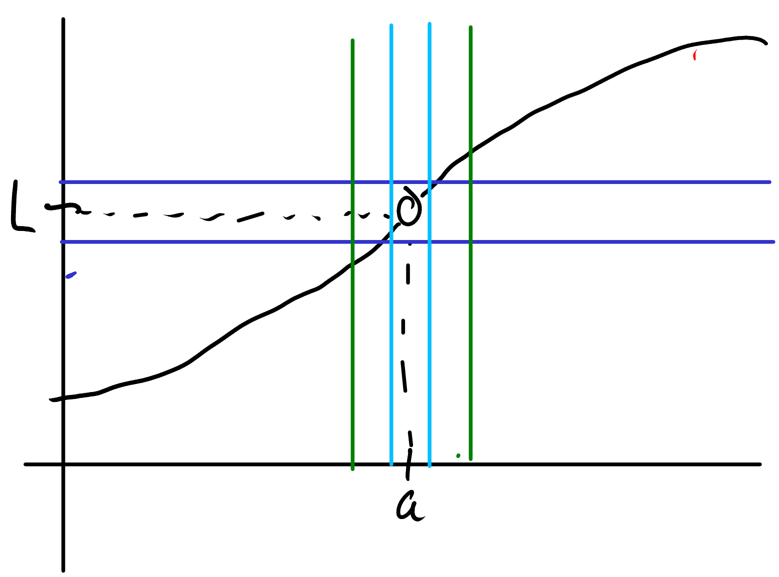


$\lim_{x \to a} f(x) = L$

Informal definition $\lim_{x \to a} f(x) = L$ means that we can make $f(x)$ as close to L as want by requiring x to be close to (but not equal to) a.



Red interval doesn't work.
green interval does work.



ε = epsilon

"f(x) is close to L" $\quad L-\varepsilon < f(x) < L+\varepsilon$

$-\varepsilon < f(x)-L < \varepsilon$

$|f(x)-L| < \varepsilon$

$$\left\{ \begin{array}{c} |a| < b \\ -b < a < b \end{array} \right\}$$

"x is close to a" $\quad a-\delta < x < a+\delta$

"but not equal to a" $\quad -\delta < x-a < \delta$

$|x-a| < \delta$

$x \neq a \Rightarrow x-a \neq 0 \Rightarrow |x-a| > 0$

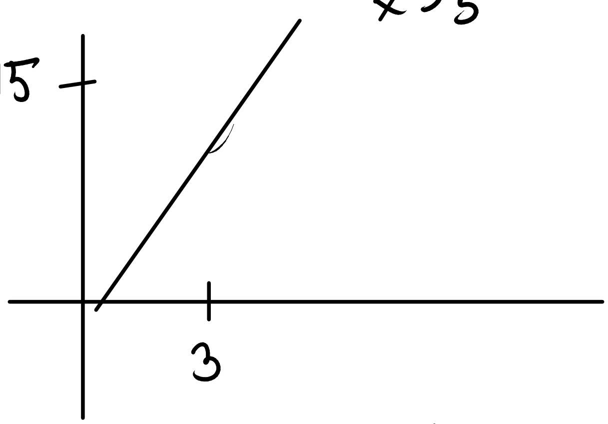
$0 < |x-a| < \delta$

Precise definition: for every $\varepsilon > 0$, there is a $\delta > 0$ so that

$0 < |x-a| < \delta$ implies $|f(x)-L| < \varepsilon$

Prove formally: $f(x) = 3x \quad a = 5$

prove $\lim_{x \to 5} 3x = 15$



Let $\varepsilon > 0$ be given

We need δ so that

$0 < |x-5| < \delta$

implies $|3x-15| < \varepsilon$

Goal inequality: $\quad |3x-15| < \varepsilon$

$|3(x-5)| < \varepsilon$

$|3|\,|x-5| < \varepsilon$

$3 \cdot |x-5| < \varepsilon$ Take $\delta = \frac{\varepsilon}{3}$

Then $|x-5| < \delta = \frac{\varepsilon}{3}$

implies $3|x-5| < \varepsilon$

and so $|3x-15| < \varepsilon$.

Prove $\lim_{x \to a} c = c$ $f(x)=c$ is a constant

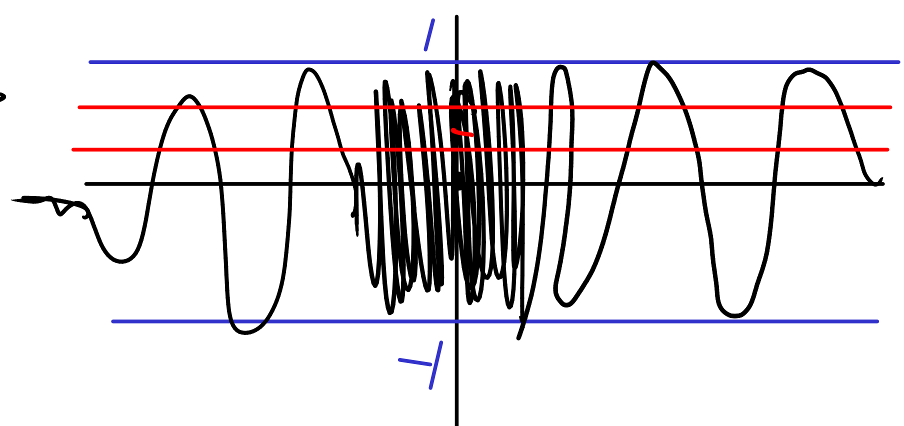
for every $\varepsilon > 0$ there is a $\delta > 0$ so that $0 < |x-a| < \delta$ implies $|f(x)-c| < \varepsilon$

But if $f(x)=c$ is constant then $|f(x)-c| = 0$

$|f(x)-c| < \varepsilon$ is automatically true, no matter what δ you pick

Take $\delta = 1$

$f(x) = \sin \frac{1}{x}$



Fact: $\lim_{x \to 0} f(x)$ does not exist.

Consequences: Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

$$\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$$

$$\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$ as long as $\lim_{x\to a} g(x) \neq 0$

Suppose $\lim_{x\to a} f(x) = 1$ $\quad \lim_{x\to a} g(x) = -1$

Find $\lim_{x\to a} \dfrac{f(x)+5g(x)}{f(x)-g(x)} = -2$

$$\lim_{x\to a} f(x)+5g(x) = 1+5(-1) = -4$$

$$\lim_{x\to a} f(x)-g(x) = 1-(-1) = 2$$

Total $= \dfrac{-4}{2} = -2$

Use product rule $\lim_{x\to a} [f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n$

$\lim_{x\to a} c = c \qquad \lim_{x\to a} x = a$

$\lim_{x \to a} x^n = a^n$ $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (require $a > 0$ if n even)

$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ (require $\lim_{x \to a} f(x) > 0$ if n is even)

Combine rules:

~~If~~ f is a polynomial $\lim_{x \to a} f(x) = f(a)$

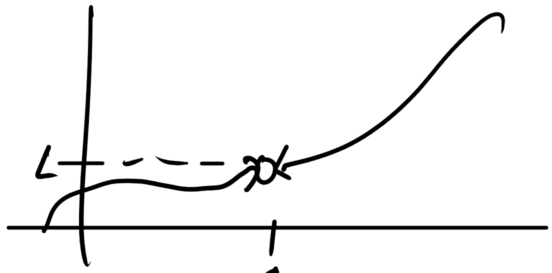
$\lim_{x \to a} x^5 + 2x^3 + 3x + 4 = a^5 + 2a^3 + 3a + 4$

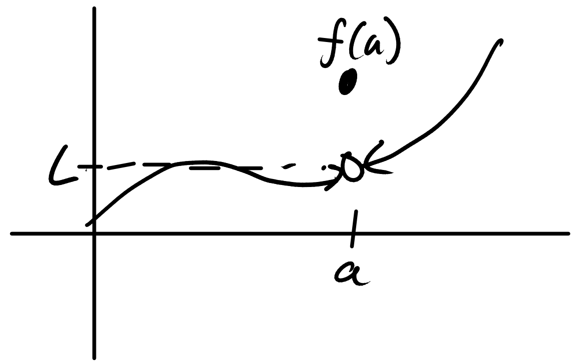
Also true for a rational function $f(x) = \frac{g(x)}{h(x)}$
where g and h are polynomials and $h(a) \neq 0$
We have $\lim_{x \to a} \frac{g(x)}{h(x)} = \frac{g(a)}{h(a)}$
Can just plug in if a is in denom. $(h(a) \neq 0)$

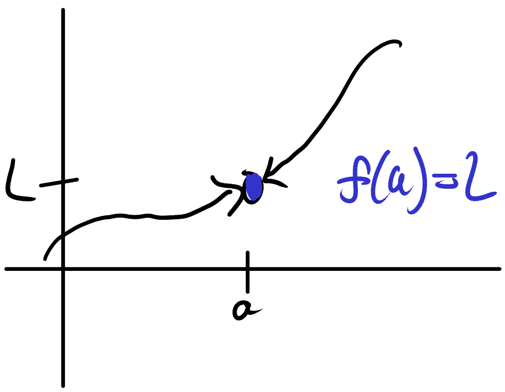
$\lim_{x \to a} f(x)$ We don't care about the value of f at a, or even if it is defined.

If $f(x) = g(x)$ for all x near a, but not necessarily at a itself

then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$







Eg. $\lim_{x\to 5} \dfrac{x^2-6x+5}{x-5}$ This function is undefined at $x=5$.

On the other hand $x^2-6x+5 = (x-1)(x-5)$

so $\dfrac{x^2-6x+5}{x-5} = \dfrac{(x-1)\cancel{(x-5)}}{\cancel{x-5}} = x-1$

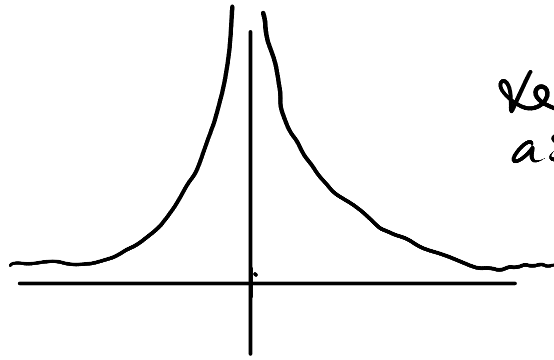
equality is valid as long as $x \neq 5$

Nevertheless conclude

$$\lim_{x\to 5} \frac{x^2-6x+5}{x-5} = \lim_{x\to 5} x-1 = 5-1 = 4$$

Infinite limits

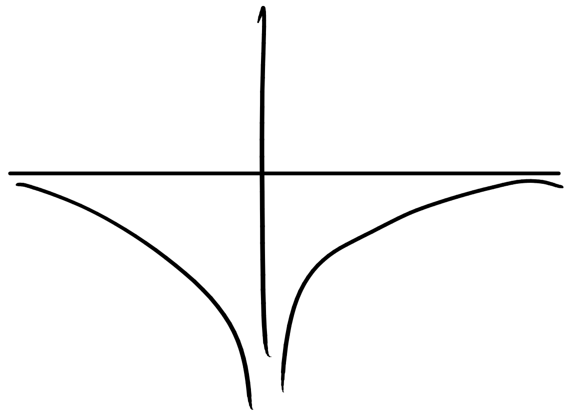
$f(x) = \frac{1}{x^2}$



vertical asymptote

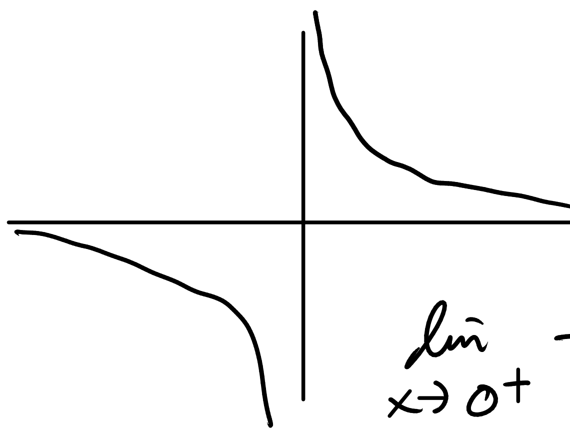
$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

$f(x) = -\frac{1}{x^4}$



$$\lim_{x\to 0} -\frac{1}{x^4} = -\infty$$

$f(x) = \frac{1}{x}$

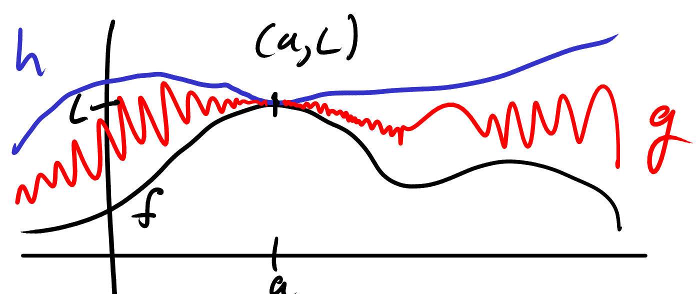


$\lim_{x\to 0} \frac{1}{x}$ Doesn't exist

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Squeeze theorem

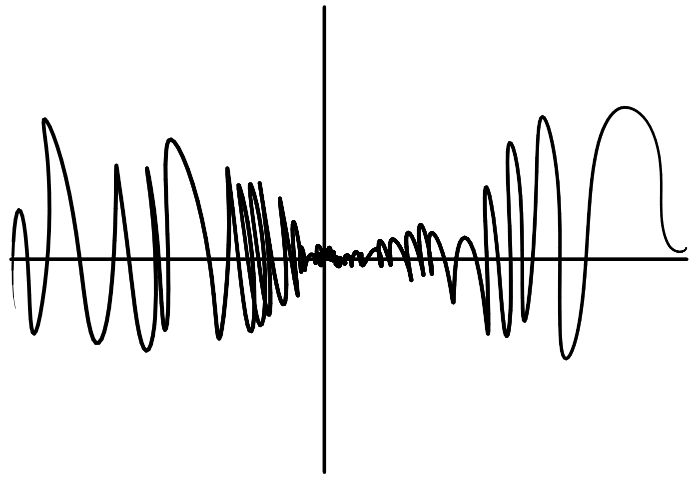


If $f(x) \le g(x) \le h(x)$ for all x near a except possibly at a itself

and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

THEN $\lim_{x \to a} g(x) = L$

$f(x) = x \sin \frac{1}{x}$



Prove

$\lim_{x \to 0} x \sin \frac{1}{x} = 0$

Notice $\left| \sin \frac{1}{x} \right| \leq 1$ $\qquad -1 \leq \sin \frac{1}{x} \leq 1$

$-x \leq x \sin \frac{1}{x} \leq x$

$\lim_{x \to 0} x = 0 \qquad \lim_{x \to 0} -x = 0$

By squeeze theorem $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ QED