Functions: Exponential, Logarithm, inverses

Exponential a^x where $a > 0$ and x is any real number.

$10^{3.14159\ldots} = ?$

Properties of a^x: (0) $a^0 = 1$ $a^1 = a$

(1) $a^{x+y} = a^x \cdot a^y$

(2) $a^{x-y} = \frac{a^x}{a^y}$ in particular $a^{-x} = \frac{1}{a^x}$

(3) $(a^x)^y = a^{xy}$

(4) $(ab)^x = a^x b^x$

$a^2 = a^{1+1} = a^1 \cdot a^1 = a \cdot a$

$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$

$(a^{1/n})^n = a^{\frac{1}{n} n} = a^1 = a \Rightarrow a^{1/n} = \sqrt[n]{a}$

Implies that $a^{n/m} = \left(\sqrt[m]{a}\right)^n$

Properties 1-4 let us compute a^x where $x = \frac{n}{m}$ n and m integers.

Can we define a^x for real x so that property Of 1 hold?

Yes, but in infinitely many ways!

The one we want is the one which is continuous as a function of x.

Physical definition of exponential function e^x

$$e = 2.718281828\ldots$$

P moves: at $t=0$ P is at $x=1$

P moves right so that the speed is equal to the distance from 0.

The position x of P from 0, after time t has elapsed, is e^t

$$\boxed{x = e^t}$$

Given x, how much time does it take for P to reach x (starting from 1)?

Answer $\boxed{t = \ln x}$

It is possible to prove $e^{t_1+t_2}=e^{t_1}e^{t_2}$.

[Calculus $P(t)$ = position at time t.

$$P(0)=1, \quad P'(t)=P(t) \Rightarrow P(t)=e^t$$]

This discussion defines e^x and $\ln x$

and we find $e^{\ln x}=x$, $\ln e^x = x$.

<u>Define</u>: $a^x=(e^{\ln a})^x = e^{(\ln a)x}$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\log_a a^x = \frac{\ln a^x}{\ln a} = \frac{\ln e^{(\ln a)x}}{\ln a}$$

$$= \frac{(\ln a)x}{\ln a} = x$$

Properties of $\log_a$ (0) $\log_a a = 1$ $\log_a 1 = 0$

$a^0=1$ $\quad \underset{\underset{0}{\|}}{\log_a a^0} = \log_a 1$

(1) $\log_a(xy) = \log_a x + \log_a y$

$$a^{\log_a x + \log_a y} = a^{\log_a x} a^{\log_a y}$$

$$= x \cdot y$$

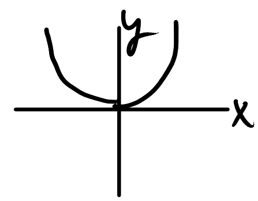
(2) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

particularly $\log_a\left(\frac{1}{y}\right) = -\log_a y$

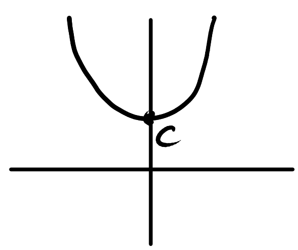
(3) $\log_a(x^b) = b \log_a x$

$$x^b = \left(a^{\log_a x}\right)^b = a^{b \log_a x}$$

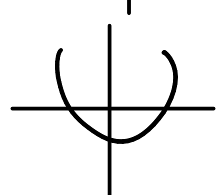
Generalities about functions $y = f(x)$



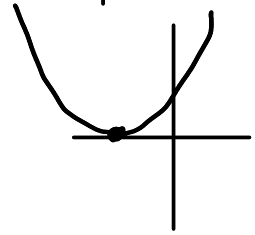
$c>0$ $f(x)+c$ shift up



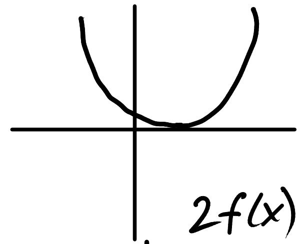
$c>0$ $f(x)-c$ shift down



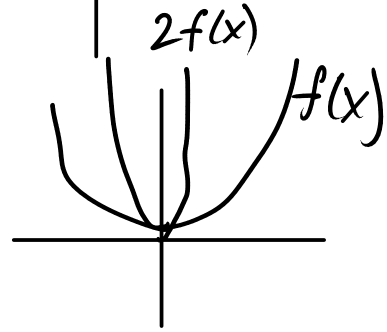
$c>0$ $f(x+c)$ shift <u>left</u>



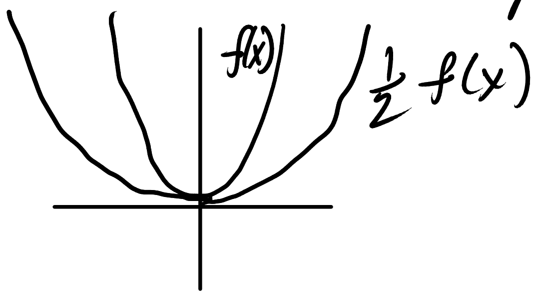
$f(x-c)$ shift right



$c>1$ $\quad cf(x)$ stretches vertically

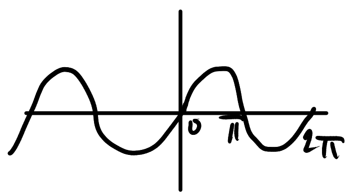


$0<c<1$ $\quad cf(x)$ shrink vertically

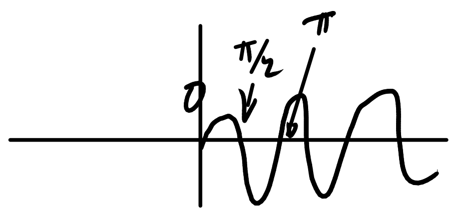


$c>1$ $\quad f(cx)$ shrinks horizontally

$f(x) = \sin x$



$f(2x) = \sin 2x$



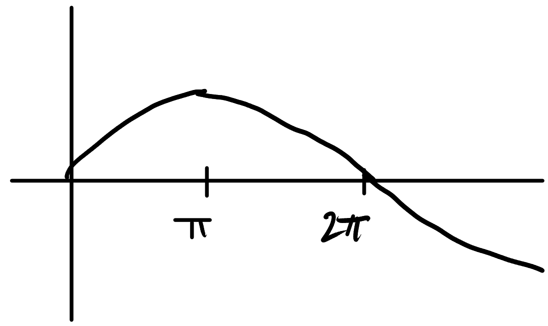
NB: make difference whether change is inside or outside of $f(\ )$.

$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$

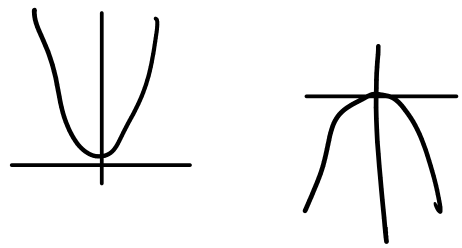
$(f \cdot g)(x) = f(x) \cdot g(x)$

$0 < c < 1$ $\quad f(cx)$ stretch horizontally

$f(\frac{1}{2}x) = \sin \frac{1}{2}x$



$-f(x)$ vertical flip



$f(-x)$ horizontal flip.

should go here.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Domain changes: forbid x such that $g(x) = 0$.

Composition $f(x)$ $g(x)$ can form

$$(f \circ g)(x) = f(g(x))$$

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

First apply h, then g, then f.

$f(x) = \sqrt{x}$ $\qquad g(x) = \sqrt{x-1}$

$$(f \circ g)(x) = \sqrt{\sqrt{x-1}} = (x-1)^{1/4}$$

$$(g \circ f)(x) = \sqrt{(\sqrt{x}) - 1}$$

$$(g \circ g \circ g)(x) = \sqrt{\left[\sqrt{(\sqrt{x-1}) - 1}\right] - 1}$$

Nested square roots

$$\sqrt{\sqrt{x-1} - 1} \geq 1$$

$$\sqrt{x-1} - 1 \geq 1$$

$$\sqrt{x-1} \geq 2$$

$$x - 1 \geq 4$$

$$x \geq 5$$

Inverse function $f^{-1}(x)$ for a given $f(x)$

means $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$

and $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$

Example $f(x) = e^x$ $\quad f^{-1}(x) = \ln x$

Example
$$\left[\begin{array}{l} f(x) = -x \qquad f^{-1}(x) = -x = f(x) \\ f(f(x)) = -(-x) = x \end{array}\right.$$

Identity function: $I(x) = x$

$(f \circ I)(x) = f(I(x)) = f(x) \qquad f \circ I = f$

similarly $I \circ f = f$

Inverse functions satisfy $f^{-1} \circ f = I$
$f \circ f^{-1} = I$

Inverse functions don't always exist!

$y = f(x)$ if and only if $x = f^{-1}(y)$

There may be two values of x : x_1 and x_2

such that $f(x_1) = y = f(x_2)$

Eg $f(x) = x^2$ $f(1) = f(-1)$

If this happens, what is $f^{-1}(y)$?

would need $f^{-1}(y) = x_1$ and $f^{-1}(y) = x_2$
which is impossible if $x_1 \neq x_2$

A function $f(x)$ is <u>one-to-one</u> if

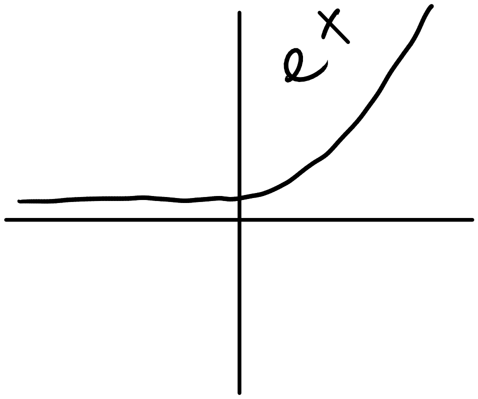
$f(x_1) = f(x_2)$ only happens when $x_1 = x_2$

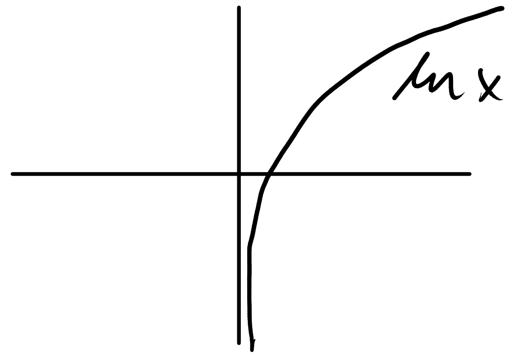
If f is one-to-one there is f^{-1}

Domain of f^{-1} = range of f
Range of f^{-1} = domain of f

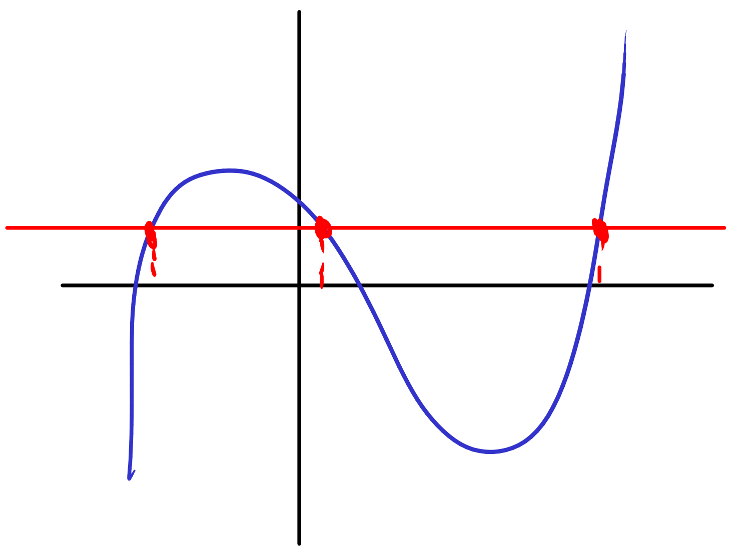
Domain of e^x = $(-\infty, \infty)$ = range of $\ln x$

Range of e^x = $(0, \infty)$ = domain of $\ln x$



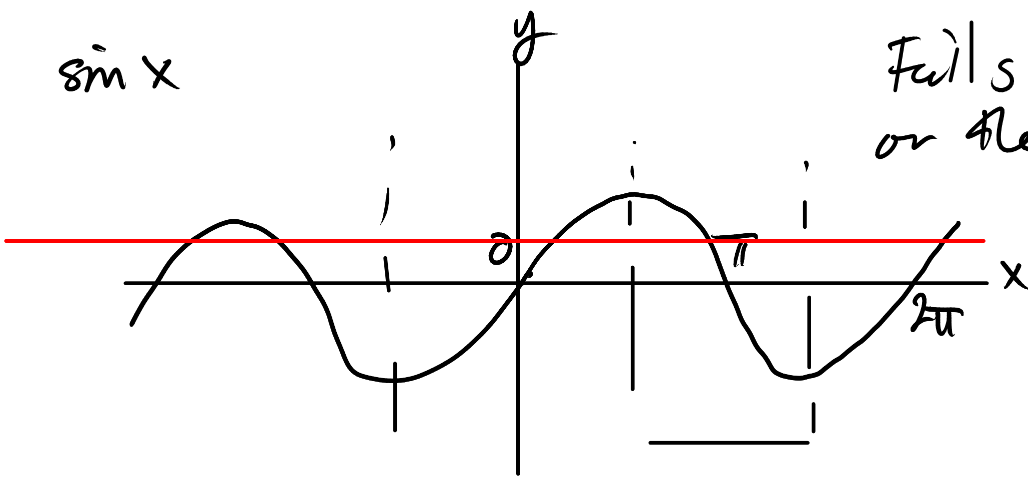


A function is one-to-one if its graph satisfies horizontal line test



Not one-to-one

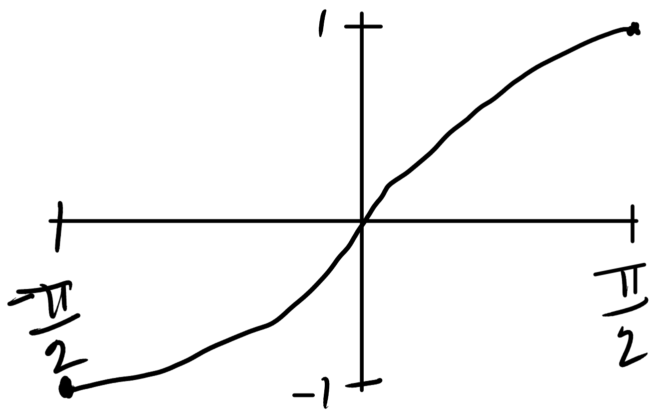
$\sin x$



Fails horiz line test on the domain $(-\infty, \infty)$

To get around this restrict the domain

Domain $= \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$



$\sin^{-1}(y)$ is the inverse function to this

Domain of $\sin^{-1}(y) = [-1, 1]$

Range of $\sin^{-1}(y) = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$