NAME AND EID: SOLUTIONS

M 408C Final Exam Free Response (version B)
December 16, 2013
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INSTRUCTIONS:

- No books, notes, calculators, or other electronic devices. Do not look at anyone else's paper during the test.
- You must show your work to receive full credit. No credit will be awarded for a mere numerical answer, even if the number is correct.
- Only answers written on these sheets will be graded. Work written on scratch paper will not be considered.

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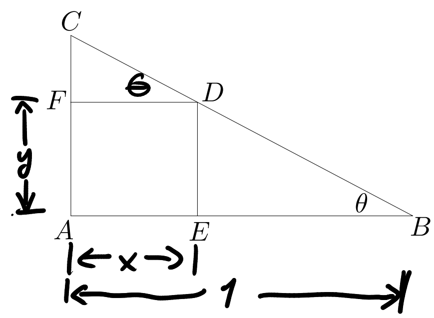
Problem	Possible	Actual
1	25	
2	45	
Total	70	

1. (25 points) A rectangle $AEDF$ is inscribed into a right triangle ABC as shown in the figure. You should assume

   - The angle of the triangle at B is **fixed** to a particular value θ.
   - The side AB has the **fixed** length 1.

   The point D is allowed to move along the side BC. (The points E and F move correspondingly so that $AEDF$ remains a rectangle.) Find the maximum area of the rectangle $AEDF$.



$x = AE$
$y = AF$

$$\tan\theta = \frac{DE}{EB} = \frac{AF}{1-AE} = \frac{y}{1-x}$$

$$\therefore \quad y = (1-x)\tan\theta$$

$$\text{Area} = xy = x(1-x)\tan\theta = (x-x^2)\tan\theta$$

Use x as the parameter: critical points are where

$$0 = \frac{d}{dx}(\text{Area}) = \frac{d}{dx}\left((x-x^2)\tan\theta\right) = (1-2x)\tan\theta$$

$$\Leftrightarrow \quad 1-2x = 0 \quad \Leftrightarrow \quad x = \frac{1}{2}$$

At this point, $\text{Area} = x(1-x)\tan\theta$

$$= \frac{1}{2}\left(1-\frac{1}{2}\right)\tan\theta = \frac{1}{2}\,\frac{1}{2}\tan\theta$$

$$= \boxed{\frac{1}{4}\tan\theta}$$

2. (45 points, 15 points per part) Below are five indefinite integrals. **You are asked to find THREE of them.** You are free to choose any three of the five problems to answer. Each part is worth 15 points, for a total of 45 points.

   If you complete more than three of the problems, you will receive up to 15 points extra credit per problem.

   (a) $\int \frac{x+3}{1+x^2}\,dx = \int \frac{x}{1+x^2}dx + 3\int \frac{1}{1+x^2}dx$

1st term: $\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|1+x^2|$

$u = 1+x^2$

$du = 2x\,dx$

2nd term $3\int \frac{1}{1+x^2}dx = 3\tan^{-1}x$

Total : $\frac{1}{2}\ln|1+x^2| + 3\tan^{-1}x + C$

   (b) $\int \sin\sqrt{x}\,dx$ substitute $u=\sqrt{x}$, $x=u^2$, $dx=2u\,du$

$\parallel$

$\int \sin u \cdot 2u\,du = 2\int u\sin u\,du$

Now use integration by parts : $v = u$ $\quad dv = du$

$dw = \sin u\,du \quad w = -\cos u$

$\int v\,dw = vw - \int w\,dv$

$2\int u\sin u\,du = -2u\cos u - 2\int -\cos u\,du$

$= -2u\cos u + 2\sin u + C$

$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

(c) $\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$

$$= \int (1-\cos^2 x) \cos^2 x \sin x \, dx \quad \left\{ \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right\}$$

$$= \int (1-u^2) u^2 (-du) = -\int (u^2 - u^4) du$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

(d) $\int \frac{x}{\sqrt{x^2-5}} \, dx \qquad u = x^2 - 5$

$du = 2x \, dx$

$$= \int \frac{\frac{1}{2} du}{\sqrt{u}} = \int \frac{1}{2} u^{-1/2} du = \frac{1}{2} \frac{u^{1/2}}{1/2} + C$$

$$= u^{1/2} + C = (x^2-5)^{1/2} + C = \sqrt{x^2-5} + C$$

Substitution $x = \sqrt{5} \sec\theta$ also works.

(e) $\int \frac{3y}{y^2 - y - 2}\, dy$ $\quad y^2 - y - 2 = (y+1)(y-2)$

Partial fractions

$$\frac{3y}{y^2-y-2} = \frac{3y}{(y+1)(y-2)} = \frac{A}{y+1} + \frac{B}{y-2}$$

$$3y = A(y-2) + B(y+1)$$

Plug in $y=-1$: $-3 = A\cdot(-3) + 0 \quad \therefore A = 1$

Plug in $y = 2$: $6 = 0 + B(3) \quad \therefore B = 2$

$$\frac{3y}{(y+1)(y-2)} = \frac{1}{y+1} + \frac{2}{y-2}$$

$$\int \frac{3y}{(y+1)(y-2)}\, dy = \int \frac{1}{y+1}\, dy + \int \frac{2}{y-2}\, dy$$

$$= \ln|y+1| + 2\ln|y-2| + C$$