NAME AND EID: SOLUTIONS

M 408C Final Exam Free Response (version A)
December 16, 2013
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INSTRUCTIONS:

- No books, notes, calculators, or other electronic devices. Do not look at anyone else's paper during the test.
- You must show your work to receive full credit. No credit will be awarded for a mere numerical answer, even if the number is correct.
- Only answers written on these sheets will be graded. Work written on scratch paper will not be considered.

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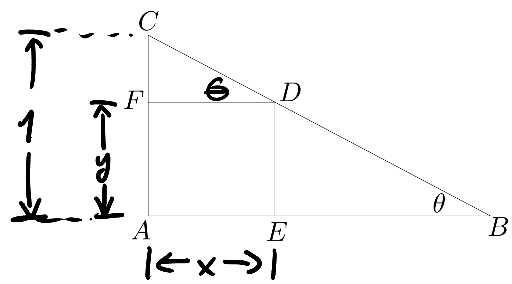
Problem	Possible	Actual
1	25	
2	45	
Total	70	

1. (25 points) A rectangle $AEDF$ is inscribed into a right triangle ABC as shown in the figure. You should assume

   - The angle of the triangle at B is **fixed** to a particular value θ.
   - The side AC has the **fixed** length 1.

   The point D is allowed to move along the side BC. (The points E and F move correspondingly so that $AEDF$ remains a rectangle.) Find the maximum area of the rectangle $AEDF$.



$x = AE$

$y = AF$

similar triangles $\Rightarrow$ angle $CDF = \theta$

$$\tan\theta = \frac{CF}{FD} = \frac{1-AF}{AE} = \frac{1-y}{x}$$

$$\therefore \quad 1-y = x\tan\theta$$

$$y = 1 - x\tan\theta$$

$$\text{Area} = xy = x(1-x\tan\theta) = x - x^2\tan\theta$$

Use x as the parameter: critical points are where

$$0 = \frac{d}{dx}(\text{Area}) = \frac{d}{dx}(x - x^2\tan\theta) = 1 - 2x\tan\theta$$

$$\Leftrightarrow 1 = 2x\tan\theta \Leftrightarrow x = \frac{1}{2\tan\theta} = \frac{1}{2}\cot\theta$$

At this point, $\text{Area} = x(1-x\tan\theta)$

$$= \frac{1}{2}\cot\theta\left(1 - \frac{1}{2}\cot\theta\tan\theta\right) =$$

$$= \frac{1}{2}\cot\theta\cdot\frac{1}{2} = \boxed{\frac{1}{4}\cot\theta}$$

2. (45 points, 15 points per part) Below are five indefinite integrals. **You are asked to find THREE of them.** You are free to choose any three of the five problems to answer. Each part is worth 15 points, for a total of 45 points.

   If you complete more than three of the problems, you will receive up to 15 points extra credit per problem.

   (a) $\int \frac{x-2}{1+x^2}\,dx$ $= \int \frac{x}{1+x^2}dx - 2\int \frac{1}{1+x^2}dx$

1st term: $\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|1+x^2|$

$u = 1+x^2$

$du = 2x\,dx$

2nd term $-2\int \frac{1}{1+x^2}dx = -2\tan^{-1}x$

Total : $\frac{1}{2}\ln|1+x^2| - 2\tan^{-1}x + C$

   (b) $\int \cos\sqrt{x}\,dx$ substitute $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$

$=$

$\int \cos u \cdot 2u\,du = 2\int u\cos u\,du$

Now use integration by parts : $v = u$ $\quad dv = du$

$dw = \cos u\,du$ $\quad w = \sin u$

$\int v\,dw = vw - \int w\,dv$

$2\int u\cos u\,du = 2u\sin u - 2\int \sin u\,du$

$= 2u\sin u + 2\cos u + C$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

(c) $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx \quad \left\{ \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right\}$$

$$= \int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

(d) $\int \frac{x}{\sqrt{x^2 - 7}} \, dx$ $\quad u = x^2 - 7$

$du = 2x \, dx$

$$= \int \frac{\frac{1}{2} du}{\sqrt{u}} = \int \frac{1}{2} u^{-1/2} \, du = \frac{1}{2} \frac{u^{1/2}}{1/2} + C$$

$$= u^{1/2} + C = (x^2 - 7)^{1/2} + C = \sqrt{x^2 - 7} + C$$

Substitution $x = \sqrt{7} \sec \theta$ also works.

(e) $\int \frac{3y}{y^2+y-2}\,dy$ $\quad y^2+y-2=(y-1)(y+2)$

Partial fractions

$$\frac{3y}{y^2+y-2} = \frac{3y}{(y-1)(y+2)} = \frac{A}{y-1} + \frac{B}{y+2}$$

$$3y = A(y+2) + B(y-1)$$

Plug in $y=1$; $3 = A \cdot 3 + 0 \quad \therefore A = 1$

Plug in $y=-2$: $-6 = 0 + B(-3) \quad \therefore B = 2$

$$\frac{3y}{(y-1)(y+2)} = \frac{1}{y-1} + \frac{2}{y+2}$$

$$\int \frac{3y}{(y-1)(y+2)}\,dy = \int \frac{1}{y-1}\,dy + \int \frac{2}{y+2}\,dy$$

$$= \ln|y-1| + 2\ln|y+2| + C$$