Nonhomogeneous Equations

General $\quad y'' + p(t)y' + q(t)y = g(t) \quad$ (Non-homog)

Has a <u>Homogeneous Version</u>

$$y'' + p(t)y' + q(t)y = 0 \quad \text{(homog)}$$

Eg. constant coefficients

$$ay'' + by' + cy = g(t) \quad \text{(Non-homog)}$$

$$ay'' + by' + cy = 0 \quad \text{(homog)}$$

Can consider Initial value problem for (Non-homog)

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0 \ , \ y'(t_0) = y_0' \end{cases}$$

General solution has two arbitrary constants

Initial conditions determine the constants.

To find general solution of $y'' + p(t)y' + q(t)y = g(t)$

PART I: find general solution of homogeneous version

$y = c_1 y_1 + c_2 y_2$ solving $y'' + p(t)y' + q(t)y = 0$

PART II: Find any particular solution $Y(t)$ to the non homogeneous equation

(Just need one particular solution)

Then $y = c_1 y_1 + c_2 y_2 + Y$ is a general solution for the non homogeneous equation.

Reason: Principle of superposition revisited

Differential operator:

Let's introduce abbreviation

$$L[y] = y'' + p(t) y' + q(t) y$$

If $y_1(t)$ is a function, $L[y_1(t)]$ is some other function.

eg. $p(t) = t$, $q(t)$ $L[y] = y'' + ty' + y$

$$L[t^3] = (t^3)'' + t \cdot (t^3)' + t^3 = 6t + 3t^3 + t^3 = 4t^3 + 6t$$

$$L[\cos t] = (\cos t)'' + t(\cos t)' + \cos t = -\cos t - t\sin t + \cos t$$
$$= -t\sin t$$

Homogeneous equation : $L[y] = 0$

Nonhomogeneous equation: $L[y] = g(t)$

$$L[y] = y'' + p(t)y' + q(t)y$$

<u>Principle of superposition</u> (L is a linear differential operator

if $u(t)$ and $v(t)$ are functions

and $L[u(t)] = g(t)$ and $L[v(t)] = h(t)$

then $L[u(t) + v(t)] = g(t) + h(t)$

ie $L[u(t)+v(t)] = L[u(t)] + L[v(t)]$

Translation into ODEs

if $u'' + p(t)u' + q(t)u = g(t)$

& $v'' + p(t)v' + q(t)v = h(t)$

then $(u+v)'' + p(t)(u+v)' + q(t)(u+v) = g(t) + h(t)$

ALSO $L[cu(t)] = cg(t)$

and $L[c_1u(t) + c_2v(t)] = c_1g(t) + c_2h(t)$

$L[c_1u(t) + c_2v(t)] = c_1L[u(t)] + c_2L[v(t)]$

<u>Facts</u>. (1) if $L[y_1] = 0$ and $L[y_2] = 0$
homog homog

then $L[c_1y_1 + c_2y_2] = 0$

Pf. $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2] = c_1 \cdot 0 + c_2 \cdot 0 = 0$

(2) if $\overset{\text{homog}}{L[y_1]=0}$ and $\overset{\text{non homog}}{L[Y]=g(t)}$

$$L[y_1+Y] = L[y_1]+L[Y] = 0+g(t) = g(t)$$

so $L[y_1+Y]=g(t)$ (y_1+Y solves non homog)

(3) if $\overset{\text{non homog}}{L[Y_1]=g(t)}$ and $\overset{\text{non homog}}{L[Y_2]=g(t)}$

$$L[Y_1-Y_2] = L[Y_1]-L[Y_2] = g(t)-g(t) = 0$$

so $L[Y_1-Y_2]=0$ (Y_1-Y_2 solve homog)

(4) If $y_h = c_1y_1+c_2y_2$ is the general solution of $L[y]=0$ and $L[Y]=g(t)$

then $y_h+Y = c_1y_1+c_2y_2+Y$ is the general solution of $L[y]=g(t)$

Proof:

$$\begin{aligned} L[c_1y_1+c_2y_2+Y] &= c_1L[y_1]+c_2L[y_2]+L[Y] \\ &= c_1\cdot 0 + c_2\cdot 0 + g(t) \\ &= g(t) \end{aligned}$$

Suppose Z is any solution of $L[y]=g(t)$

then $L[Z-Y]=0$

so $Z-Y$ solves $L[y]=0$ and hence

$Z-Y = c_1 y_1 + c_2 y_2$ for some constants c_1, c_2

$Z = c_1 y_1 + c_2 y_2 + Y$

Now constant coefficient linear nonhomogeneous

First method: Undetermined coefficients
(educated guessing)

$$L[y] = y'' - 2y' - y = \begin{cases} 0 & \text{homog} \\ 3e^{2t} & \text{nonhomog} \end{cases}$$

To find general solution need to
(1) solve homog version → see previous lectures
(2) find just one solution of the nonhomog version.

Try something with e^{2t} in it $Y = Ae^{2t}$

$$L[Ae^{2t}] = (Ae^{2t})'' - 2(Ae^{2t})' - (Ae^{2t})$$

$$= A \cdot 4e^{2t} - 2 \cdot A \cdot 2e^{2t} - Ae^{2t} = -Ae^{2t}$$

Need $-Ae^{2t} = 3e^{2t}$ $\quad A = -3$

so $y = -3e^{2t}$ solves $L[y] = 3e^{2t}$
non homog.

If you solve the homogeneous version:

$y_1 = e^{(1+\sqrt{2})t} \qquad y_2 = e^{(1-\sqrt{2})t}$

General non homogeneous solution

$y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t} - 3e^{2t}$

Could have a different particular solution

But get same general solution by renaming the constants.

<u>Example</u>: $y'' - 2y' - y = e^t \cos(2t)$

Try $y = Ae^t\cos(2t) + Be^t\sin(2t)$ (generally need both sin & cos)

$y' = (A+2B)e^t\cos(2t) + (-2A+B)e^t\sin(2t)$

$y'' = (-3A+3B)e^t\cos(2t) + (-4A-3B)e^t\sin(2t)$

$L[y] = y'' - 2y' - y$

$= (-6A-B)e^t\cos(2t) + (0\cdot A - 6B)e^t\sin(2t)$

Need $= e^t \cos(2t)$

$$-6A - B = 1 \qquad -6B = 0$$

$$A = -\frac{1}{6} \quad \Leftarrow \quad B = 0$$

$$Y = -\frac{1}{6} e^t \cos(2t)$$

general solution (nonhomogeneous) $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t} - \frac{1}{6} e^t \cos(2t)$

Another view point:

$$L[y] = ay'' + by' + cy = (aD^2 + bD + c)y$$

where $D = \frac{d}{dt} = (\ )'$ is the operator for differentiation.

$$L[e^{rt}] = (aD^2 + bD + c)e^{rt}$$
$$= (ar^2 + br + c)e^{rt}$$

$$De^{rt} = re^{rt}$$
$$(D-r)e^{rt} = 0$$

<u>Ex</u> $L[y] = y'' - 2y' - y = 8e^{14t}$

$$L[y] = (D^2 - 2D - 1)y$$

Try $Y = Ae^{14t}$

$$8e^{14t} = L[Ae^{14t}] = A\,L[e^{14t}] = ((14)^2 - 2\cdot 14 - 1)\,Ae^{14t}$$

$$L[e^{rt}]\,(ar^2+br+c)e^{rt}$$

$$\Big(\underbrace{(14)^2-2\cdot 14-1}_{=\,167}\Big)Ae^{14t} = 8e^{14t}$$

$$A = \frac{8}{167} \qquad y = \frac{8}{167}e^{14t}$$

<u>Ex</u> $\overset{L[y]}{\overbrace{y''-2y'-y}} = \sin t = \frac{1}{2i}\left(e^{it}-e^{-it}\right)$

Try $y = Ae^{it}+Be^{-it}$

$$L[e^{it}] = (i^2-2i-1)e^{it} = (-2-2i)e^{it}$$

$$L[e^{-it}] = \left((-i)^2-2(-i)-1\right)e^{-it} = (-2+2i)e^{-it}$$

$$A[-2-2i]e^{it}+B[-2+2i]e^{-it} = \frac{1}{2i}\left(e^{it}-e^{-it}\right)$$

$$\begin{cases} A[-2-2i] = \frac{1}{2i} \\ B[-2+2i] = -\frac{1}{2i} \end{cases} \qquad A = \frac{1+i}{8} \qquad B = \frac{1-i}{8}$$

$$y = \frac{1}{8}\left[(1+i)e^{it}+(1-i)e^{-it}\right]$$

Use Euler's formula to get sin and cos.

$$\boxed{\text{Table 3.5.1 p 181 for inspiration}}$$