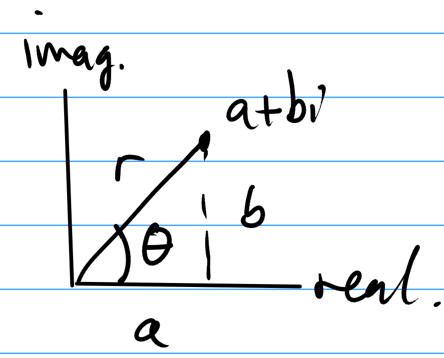
Complex roots cont. & Repeated roots

Complex numbers $a+bi$ $\quad a,b$ real

$i^2 = -1$



$a+bi$ rectangular form

$re^{i\theta}$ polar form

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. $\quad \frac{d}{dt} e^{it} = ie^{it}$

<u>Example</u> $\quad y'' + y = 0$

if e^{rt} is a solution $\quad r^2+1=0 \Rightarrow r = i$
or $r=-i$

Complex solutions $y_1 = e^{it}$, $y_2 = e^{-it}$

$u(t) = \frac{1}{2}(y_1+y_2) = \frac{1}{2}(e^{it}+e^{-it})$

$= \frac{1}{2}(\cos t + i\sin t + \cos(-t) + i\sin(-t))$

$= \frac{1}{2}(\cos t + i\sin t + \cos t - i\sin t)$

$= \frac{1}{2}(\cos t + \cos t) = \cos t$

$$v(t) = \frac{1}{2i}(y_1 - y_2) = \sin t$$

so $\cos t$ and $\sin t$ are a set of real solutions.

Idea behind these combinations:

Suppose $z = a + bi$ is a complex number.

a is called the REAL PART

b is called the IMAGINARY PART

$\bar{z} = a - bi$ is called the COMPLEX CONJUGATE

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(a + bi + a - bi) = \frac{1}{2}(a + a) = a$$

$$\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}(a + bi - a + bi) = \frac{1}{2i}(2ib) = b$$

$$\text{Real Part}(z) = \frac{1}{2}(z + \bar{z})$$

$$\text{Imag. Part}(z) = \frac{1}{2i}(z - \bar{z})$$

Fact: complex conjugates in polar form:

$z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$

Exercise: prove this using eulers formula.

Example for when roots of characteristic equation are complex

$$y'' - 2y' + 5y = 0 \qquad \text{ODE}$$

$$r^2 - 2r + 5 = 0 \qquad \text{characteristic equation.}$$

Quadratic formula $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a = 1, \ b = -2, \ c = 5$

$$r = \frac{2 \pm \sqrt{4 - 4\cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

$r_1 = 1 + 2i \qquad r_2 = 1 - 2i$

NOTICE r_1 and r_2 are complex conjugates.

$$y_1 = e^{r_1 t} = e^{(1+2i)t} = e^{t + 2it} = e^t e^{2it}$$

$$= e^t(\cos(2t) + i\sin(2t))$$

$$y_2 = e^{r_2 t} = e^{(1-2i)t} = e^t e^{-2it} = e^t e^{-2it}$$

$$= e^t(\cos(-2t) + i\sin(-2t))$$

$$= e^t(\cos(2t) - i\sin(2t))$$

Notice y_1 and y_2 are conjugates.

Real solutions

$$u(t) = \frac{1}{2}(y_1 + y_2) = e^t \cos(2t)$$

$$\quad \| \qquad\qquad\qquad \|$$

$$\frac{1}{2}(y_1 + \overline{y_1}) = \text{Real Part}(y_1)$$

$$v(t) = \frac{1}{2i}(y_1 - y_2) = e^t \sin(2t)$$

$$\quad \| \qquad\qquad\qquad \|$$

$$\frac{1}{2i}(y_1 - \overline{y_1}) = \text{Imag. Part}(y_1)$$

General Real solution: $y = c_1 u + c_2 v$

$$y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$$

Can also solve IVP $y(t_0) = y_0$, $y'(t_0) = y_0'$

Summary: $ay'' + by' + cy = 0$

char eqn : $ar^2 + br + c = 0$

If $b^2 - 4ac < 0$, then the roots are a complex conjugate pair $\left\{ \begin{matrix} \alpha + \beta i \\ \alpha - \beta i \end{matrix} \right\}$

$$\alpha \pm \beta i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

complex solutions: $y_1 = e^{(\alpha+\beta i)t} = e^{\alpha t} e^{i\beta t}$

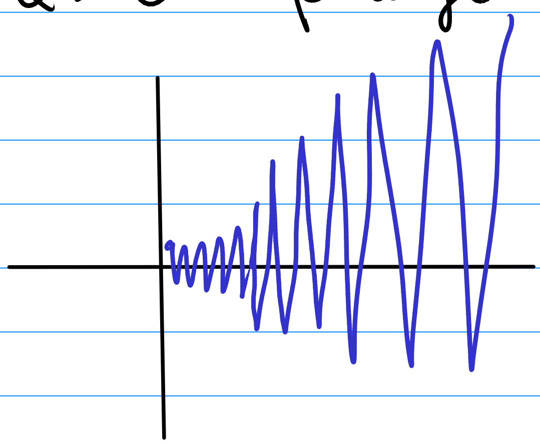
$y_2 = e^{(\alpha-\beta i)t} = e^{\alpha t} e^{-i\beta t}$

Real solutions: $u(t) = \text{Real}(y_1) = e^{\alpha t}\cos(\beta t)$

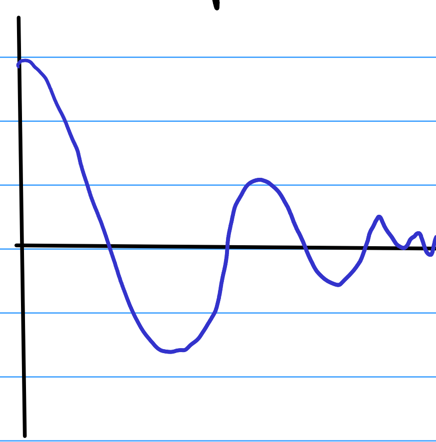
$v(t) = \text{Imag}(y_1) = e^{\alpha t}\sin(\beta t)$

Shape of solutions: $\cos(\beta t)$, $\sin(\beta t)$ oscillate while $e^{\alpha t}$ grows or decays:

$\alpha > 0$ β large



$\alpha < 0$ β small



Last case: Only one real root (repeated root)

$ay'' + by' + cy = 0$ What happens is $b^2 - 4ac = 0$

$ar^2 + br + c = 0$ $r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-b}{2a}$

in this case $ar^2+br+c = a\left(r-\left(\frac{-b}{2a}\right)\right)^2$

Still, we know that $y_1 = e^{rt} = e^{-bt/2a}$

is a solution:

But that's not enough:

general solution should look like $c_1y_1 + c_2y_2$

where y_1 and y_2 form a fundamental set.

$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1' \neq 0$

Q: What is the other solution?

Technique: reduction of order:

Idea $y_1 = e^{r_1 t} = e^{-bt/2a}$ is a solution

so Cy_1 is also a solution if C is constant.

D'alembert suggests try $v(t)y_1 = y_2$

Know y_1 is a solution want y_2 to be a solution.

$y_2' = v'y_1 + vy_1'$

$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + 2v'y_1' + vy_1''$

$$0 \overset{?}{=} ay_2'' + by_2' + cy_2 = av''y_1 + 2av'y_1' + avy_1'' + bv'y_1 + bvy_1' + cvy_1$$

$$= av''y_1 + 2av'y_1' + bv'y_1 + v\underbrace{(ay_1'' + by_1' + cy_1)}_{0 \text{ since } y_1 \text{ is solution}}$$

Now use the fact that $y_1 = e^{rt} = e^{(-b/2a)t}$

$$y_1' = \frac{-b}{2a} e^{(-b/2a)t}$$

so $2av'y_1' + bv'y_1 = 2av'\left(\frac{-b}{2a}\right)e^{rt} + bv'e^{rt}$

$$= v'(-b)e^{rt} + bv'e^{rt} = 0$$

$$= av''y_1 = ae^{\left(\frac{-b}{2a}\right)t} v'' \overset{?}{=} 0$$

Need $v'' = 0$

General solution $v(t) = c_1 t + c_2$

So $y_2 = (c_1 t + c_2) y_1 = (c_1 t + c_2) e^{\left(\frac{-b}{2a}\right)t}$

is always a solution for constants c_1 and c_2

$$y_2 = c_1 t e^{\left(\frac{-b}{2a}\right)t} + c_2 e^{\left(\frac{-b}{2a}\right)t}$$

The other solution we're looking for is

$$y = te^{\left(\frac{-b}{2a}\right)t}$$

Fundamental set $y_1 = e^{\left(\frac{-b}{2a}\right)t}$, $y_2 = te^{\left(\frac{-b}{2a}\right)t}$

<u>Example</u>: $y'' - 6y' + 9y = 0$

$$(r-3)^2 = r^2 - 6r + 9 = 0$$

$r = 3$ is the only root.

$y_1 = e^{3t}$ is a solution.

Try $y_2 = v(t)e^{3t}$

$$y_2'' - 6y_2' + 9y_2$$

$$\left.\begin{aligned} &= v''e^{3t} + 6v'e^{3t} + 9ve^{3t} \\ &-6v'e^{3t} - 18ve^{3t} \\ &+9ve^{3t} \end{aligned}\right\} = v''e^{3t} \stackrel{?}{=} 0$$

$$v = c_1 t + c_2$$

General solution: $y = c_1 te^{3t} + c_2 e^{3t}$

Fundamental set e^{3t}, te^{3t}