Continue second order linear equations.

Wronskian determinant

Complex roots of characteristic equation.

Wronskian: Consider IVP $\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0,\ y'(t_0) = y_0' \end{cases}$

e.g. $y'' - y = 0 \rightarrow$ two solutions $y_1(t) = e^t$ $y_2(t) = e^{-t}$

Suppose y_1 and y_2 are two particular sol$^{\underline{n}}$s to $y'' + p(t)y' + q(t)y = 0$

Q: can we find a combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$ that satisfies

the initial conditions $y(t_0) = y_0,\ y'(t_0) = y_0'$

That is need to solve for c_1 and c_2:

$$y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

$$y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

Note: it's two equations in two unknowns.
Good.

Bad: $x + y = 2$

inconsistent $-3x - 3y = 2$ } two equations, two unknowns

$(-3)(x+y) = 2$

No solutions

redundant $x + y = 2$

$-3x - 3y = -6$ } infinitely many solutions

Consider equations $\begin{cases} ax + by = m \\ cx + dy = n \end{cases}$

If $ad - bc \neq 0$, then this system has a unique solution.

$$ad - bc = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$ DETERMINANT

in the second order linear case,

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

If $y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0) \neq 0$ then

the system has a unique solution:

The expression

$y_1(t_0)\,y_2'(t_0) - y_2(t_0)\,y_1'(t_0) =: W(y_1,y_2)(t_0)$

is called the Wronskian of the solutions y_1 and y_2 at the point t_0.

- If y_1 and y_2 are particular solutions and $W(y_1,y_2)(t_0) \neq 0$, then the IVP $y(t_0)=y_0$, $y'(t_0)=y_0'$ always has a unique solution of the form

  $y = c_1 y_1 + c_2 y_2$

- In this case, y_1 and y_2 are called a fundamental set of solutions and $y = c_1 y_1 + c_2 y_2$ is called the general sol$^{\underline{n}}$.

Check if $ay'' + by' + cy = 0$

characteristic equation $ar^2 + br + c = 0$

solutions r_1 and r_2 real and not equal

$y_1 = e^{r_1 t}$ $\quad y_2 = e^{r_2 t}$

Wronskian: $\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix}$

$= e^{r_1 t}(r_2 e^{r_2 t}) - e^{r_2 t}(r_1 e^{r_1 t})$

$= (r_2 - r_1) e^{(r_1+r_2)t} \neq 0$ because $r_1 \neq r_2$

So $e^{r_1 t}$ and $e^{r_2 t}$ are a fundamental set of solutions.

If $r_1 = r_2$ e.g. $(r+1)^2 = r^2 + 2r + 1$

$y'' + 2y' + y = 0$

$y_1 = e^{-t}$

$y_2 = e^{-t}$ $\xrightarrow{\text{Magic}}$ te^{-t} is also a solution.

Not enough solutions

$W(y_1, y_2)(t) = 0$

Another case: Characteristic equation has complex roots.

Ex $y'' + y = 0$ $\quad r^2 + 1 = 0$ $\quad r_1 = \sqrt{-1} = i$

$r_2 = -\sqrt{-1} = -i$

i = imaginary unit $\quad i^2 = -1$

general complex number $a+bi$, where a and b are real numbers.

ADD $(1+2i)+(12-32i) = (1+12)+(2-32)i$

$= 13 + (-30)i$

MULTIPLY $(1+2i)(12-32i) \xrightarrow{\text{FOIL}} 1\cdot 12 + 1\cdot(-32)i$

$2\cdot 12 i + (2)\cdot(-32)\, i^2$

$= 12 + (-32)i + 24i - 64 i^2$

$= 12 + (24-32)i - 64(-1)$

$= 12 + 64 - 8i = 76 - 8i$

DIVISION $\dfrac{2+i}{1-3i} = \dfrac{2+i}{1-3i} \cdot \dfrac{1+3i}{1+3i}$

$1+3i$ is the complex conjugate of $1-3i$

$= \dfrac{(2+i)(1+3i)}{(1-3i)(1+3i)} = \dfrac{(2+i)(1+3i)}{1^2-(3i)^2}$

$a-bi$ is the complex conjugate of $a+bi$

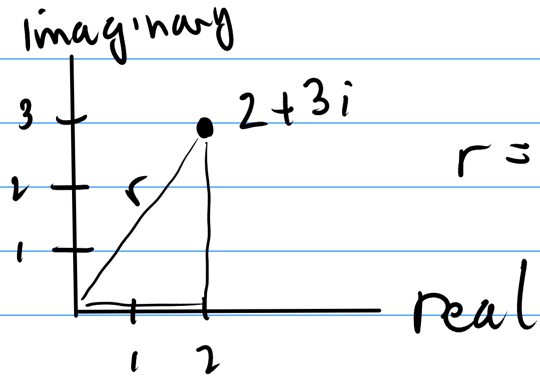
$= \dfrac{(2+i)(1+3i)}{1-(-9)} = \dfrac{(2+i)(1+3i)}{10}$

$= \dfrac{-1+7i}{10} = \dfrac{-1}{10} + \dfrac{7}{10} i$

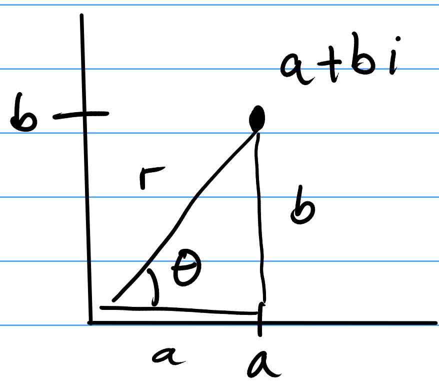
Geometry of complex numbers:

Complex number $\longleftrightarrow$ point in the plane

$a+bi$ $\qquad\qquad$ (a, b)



$$r = \sqrt{2^2+3^2} = \sqrt{13}$$



$$r = \sqrt{a^2+b^2}$$

$$\cos\theta = \frac{a}{r}$$

$$\sin\theta = \frac{b}{r}$$

$$(a,b) = (r\cos\theta, \ r\sin\theta)$$ Polar coordinates

$$a+bi = r\cos\theta + i\,r\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$ polar representation of a complex number.

<u>Definition</u>: $e^{i\theta} := \cos\theta + i\sin\theta$ Euler's formula

Why make this definition?

① Exponential law: $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i\theta_1 + i\theta_2} = e^{i(\theta_1+\theta_2)}$

② Derivative of exponential $\frac{d}{dt}\left(e^{rt}\right) = re^{rt}$

still true $\frac{d}{dt}\left(e^{it}\right) = ie^{it}$

③ Power series representations

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

$$= \cos x + i \sin x$$

<u>Prove</u>: $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$$(\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2)$$

$$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$

$$+ (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\,i$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

Because of trig identities.

$$= e^{i(\theta_1+\theta_2)}$$

Done.

Prove $\frac{d}{dt}\left(e^{it}\right) = ie^{it}$

- $\frac{d}{dt}(\cos t + i\sin t) = \frac{d}{dt}(\cos t) + \frac{d}{dt}(i\sin t)$

$$= \frac{d}{dt}(\cos t) + i\frac{d}{dt}(\sin t)$$

$$= -\sin t + i\cos t$$

- $ie^{it} = i(\cos t + i\sin t) = i\cos t + i^2\sin t$

$$= -\sin t + i\cos t$$

Done.

- $e^{5+3i} := e^5 \cdot e^{3i} = e^5(\cos 3 + i\sin 3)$
- $\frac{d}{dt}\left(e^{ibt}\right) = ibe^{ibt}$

Ex $y'' + y = 0$ if e^{rt} is a solution, then

$$(e^{rt})'' + (e^{rt}) = r^2e^{rt} + e^{rt} = (r^2+1)e^{rt}$$

$$r^2 + 1 = 0 \Rightarrow r_1 = i$$

$$r_2 = -i$$

$y_1(t) = e^{it}$ $y_2(t) = e^{-it}$

complex-valued solutions: no physical meaning.

$$y_1(t) = e^{it} = \cos t + i\sin t$$
$$y_2(t) = e^{-it} = \cos(-t) + i\sin(-t)$$
$$= \cos t + -i\sin t$$

Equation is linear $\Rightarrow$ add or subtract solutions.

$$y_1(t) + y_2(t) = \cos t + i\sin t + \cos t - i\sin t$$
$$= 2\cos t \quad \leftarrow \text{real solution.}$$

$$(2\cos t)'' + (2\cos t) = 0 \quad \checkmark$$

$$y_1(t) - y_2(t) = \cos t + i\sin t - \cos t + i\sin t$$
$$= 2i\sin t$$

$$\frac{y_1(t) - y_2(t)}{i} = 2\sin t \quad \leftarrow \text{real solution.}$$

So in fact $2\cos t$ and $2\sin t$ are a fundamental set of real solutions.

and the general solution is

$$y(t) = c_1(2\cos t) + c_2(2\sin t)$$