Second Order Linear ODE

general form: $a(t)y'' + b(t)y' + c(t)y = g(t)$

Standard form: $y'' + p(t)y' + q(t)y = g(t)$

[General nonlinear $y'' = f(t, y, y')$]

Initial Value Problem at time t_0

$$y(t_0) = y_0 \quad , \quad y'(t_0) = y_0'$$

→ Roughly speaking, solving a second order equation involves "integrating twice";

→ hence there are two constants of integration that need to be determined.

Important special case: homogeneous (RHS = 0)

$$a(t)y'' + b(t)y' + c(t)y = 0 \leftarrow$$ zero here means

or $$y'' + p(t)y' + q(t)y = 0 \leftarrow$$ HOMOGENEOUS

[independent variable is t]

Consider $y'' - y = 0$

CONSTANT COEFFICIENT LINEAR HOMOGENEOUS

$y_1(t) = e^t$

$y_2(t) = e^{-t}$

$y_2'(t) = -e^{-t}$, $y_2''(t) = (-1)^2 e^{-t} = e^{-t}$

$y_3(t) = -\sin t \quad y_3' = -\cos t \quad y_3'' = (-1)^2 \sin t = \sin t$

solves $y'' + y = 0$

Find some solutions for $y'' + 3y' + 2y = 0$

Let's try $y = e^{rt}$ and see if it solves.

$y' = re^{rt} \qquad y'' = r^2 e^{rt}$

$0 = y'' + 3y' + 2y = r^2 e^{rt} + 3re^{rt} + 2e^{rt}$

$0 = (r^2 + 3r + 2)e^{rt}$

In order to be a solution, need $\boxed{r^2 + 3r + 2 = 0}$

$0 = r^2 + 3r + 2 = (r+1)(r+2)$ CHARACTERISTIC EQUATION

$\Rightarrow$ either $r = -1$ or $r = -2$.

So $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$

are solutions.

→ Consider general constant coefficient homogeneous linear second order ODE

$$ay'' + by' + cy = 0$$

Try: e^{rt}, find that if it is a solution

$$0 = (ar^2 + br + c)e^{rt}$$

$$\Rightarrow \boxed{ar^2 + br + c = 0}$$ Characteristic equation.

What do the roots of $ar^2 + br + c = 0$ look like: there are two solutions:

→ both real and distinct eg. $(r+1)(r+2) = 0$
→ both real and equal eg. $(r+3)^2 = 0$
→ both complex and conjugate to each other

eg. $r^2 + 1 = 0$

$$\Rightarrow r = \sqrt{-1} = i$$
$$r = -\sqrt{-1} = -i$$

Today we only consider case when roots are real and distinct.

Return to $y'' - y = 0$ $\quad y_1(t) = e^t \quad y_2(t) = e^{-t}$

Other solutions?

Key idea for linear equations:

SUPERPOSITION PRINCIPLE

If y_1 and y_2 are solutions to

a homogeneous linear equation, $y'' + p(t)y' + q(t)y = 0$

Then so is $c_1 y_1 + c_2 y_2$ for any choice

of constants c_1 and c_2.

[$c_1 y_1 + c_2 y_2$ is called a linear combination of y_1 and y_2]

Eg. $y'' - y = 0$ $\quad y_1 = e^t \quad y_2 = e^{-t}$

$$y_3 = c_1 e^t + c_2 e^{-t}$$

$$y_3' = c_1 e^t - c_2 e^{-t}$$

$$y_3'' = c_1 e^t + c_2 e^{-t}$$

$$y_3'' - y_3 = 0$$

Proof of superposition

ODE $y'' + p(t)y' + q(t)y = 0$

suppose y_1 and y_2 are solutions.

Consider $c_1 y_1 + c_2 y_2 = y_3$

$y_3' = c_1 y_1' + c_2 y_2'$

$y_3'' = c_1 y_1'' + c_2 y_2''$

$y_3'' + p y_3' + q y_3$ ⇐ want to show $= 0$

$= c_1 y_1'' + c_2 y_2'' + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2)$

$= c_1 y_1'' + c_1 p y_1' + c_1 q y_1 + c_2 y_2'' + c_2 p y_2' + c_2 q y_2$

$= c_1(y_1'' + p y_1' + q y_1) + c_2(y_2'' + p y_2' + q y_2)$

$= c_1 \underbrace{(\ 0\ )}_{\text{b/c } y_1 \text{ is sol}^{\underline{n}}} + c_2 \underbrace{(\ 0\ )}_{\text{b/c } y_2 \text{ is a sol}^{\underline{n}}.}$

$= 0$ QED

Use superposition to solve initial value problem

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$

Ex $\begin{cases} y'' - y = 0 \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$

We know that $c_1 e^t + c_2 e^{-t}$ is a solution for any choice of constants c_1 and c_2.

Idea: choose c_1 and c_2 so that $y(0) = 2$ and $y'(0) = -1$

$$y(0) = c_1 e^0 + c_2 e^{-0} = c_1 + c_2 = 2$$

$$y'(t) = c_1 e^t - c_2 e^{-t}$$

$$y'(0) = c_1 e^0 - c_2 e^0 = c_1 - c_2 = -1$$

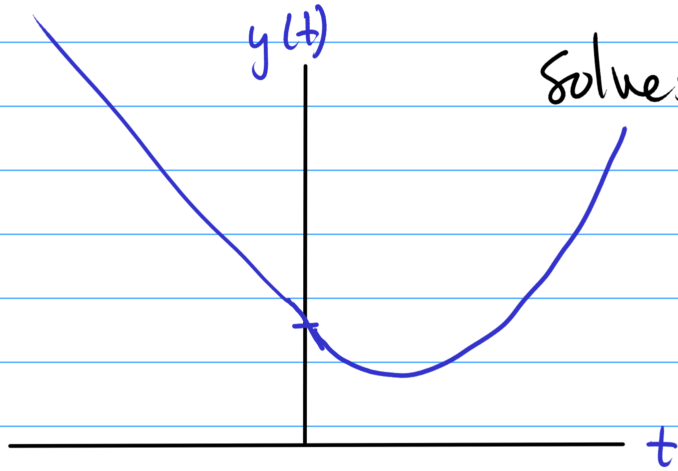
Constants need to solve a 2×2 linear system

$$\left\{ \begin{array}{l} c_1 + c_2 = 2 \\ c_1 - c_2 = -1 \end{array} \right\} \quad \text{Add} \Rightarrow 2c_1 = 1$$

$$c_1 = \tfrac{1}{2}$$

Plug in $\frac{1}{2} + c_2 = 2 \Rightarrow c_2 = \frac{3}{2}$

$c_1 = \frac{1}{2} \quad c_2 = \frac{3}{2}$ $\quad y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-t}$



solves the IVP

Ex $\begin{cases} y'' + y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$

Step 1: write down and solve characteristic equation.

Looking for e^{rt} to be a solution so need

$$\left.\begin{array}{c} r^2 + r - 2 = 0 \\ \| \\ (r+2)(r-1) \end{array}\right\} \Rightarrow \begin{array}{l} r = -2 \\ \text{or } r = 1 \end{array}$$

means that e^t and e^{-2t} solve the ODE

GENERAL SOLUTION: $y(t) = c_1 e^t + c_2 e^{-2t}$

Step 2: Use initial conditions to solve for the constants c_1 and c_2

$$y(t) = c_1 e^t + c_2 e^{-2t} \qquad y(0) = c_1 + c_2 = 1$$

$$y'(t) = c_1 e^t - 2c_2 e^{-2t} \qquad y'(0) = c_1 - 2c_2 = 2$$

$$\begin{Bmatrix} c_1 + c_2 = 1 \\ c_1 - 2c_2 = 2 \end{Bmatrix} \qquad c_1 = 1 - c_2$$

plug in:

$$(1 - c_2) - 2c_2 = 2$$

$$1 - 3c_2 = 2$$

$$-3c_2 = 1$$

$$c_2 = -\frac{1}{3}$$

$$c_1 = 1 - c_2 = 1 - \left(-\frac{1}{3}\right)$$

$$c_1 = \frac{4}{3}$$

Solution: $$\underline{\underline{y(t) = \frac{4}{3} e^t - \frac{1}{3} e^{-2t}}}$$