Exact equations & Euler's Method

Recall separable equation $\frac{dy}{dx} = F(x)\,G(y)$

Another way to write it $M(x) + N(y)\frac{dy}{dx} = 0$

$\left[M(x) = -F(x) \quad N(y) = \frac{1}{G(y)} \right]$

Integrate $H_1(x) = \int M(x)\,dx$

$H_2(y) = \int N(y)\,dy$

Find $\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = M(x) + N(y)\frac{dy}{dx}$

So if $y(x)$ is a solution: $\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = 0$

that is, $\underbrace{H_1(x) + H_2(y) = \text{constant}}$

"determines" the solution implicitly.

$\Rightarrow$ try to solve for y.

Suppose we really want a function $F(x,y)$ such that the solutions of our ODE are implicitly given by $F(x,y) = \text{constant}$.

If that's true: Assume $F(x,y) = \text{constant}$

$$\frac{d}{dx}\left[F(x,y)\right] = \frac{d}{dx}\left[\text{constant}\right] = 0$$

$$\parallel$$

$$\frac{\partial F}{\partial x}(x,y)\frac{dx}{dx} + \frac{\partial F}{\partial y}(x,y)\frac{dy}{dx}$$

$$\boxed{\frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,y)\frac{dy}{dx} = 0}$$

Want to go from the ODE to F.

Eg. Consider $y\sin x + x^2e^y - y = C$

$$\frac{d}{dx}\left[y\sin x + x^2e^y - y\right] = 0$$

$$\parallel$$

$$\frac{dy}{dx}\sin x + y\cos x + 2xe^y + x^2e^y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\left(y\cos x + 2xe^y\right) + \left(\sin x + x^2e^y - 1\right)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(y\cos x + 2xe^{y})}{(\sin x + x^2 e^{y} - 1)}$$

Consider $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

Want a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \text{ and } \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

It may not be possible to solve for F:

Example: $2y + x\frac{dy}{dx} = 0$

Want: $\begin{cases} \frac{\partial F}{\partial x} = 2y \\ \frac{\partial F}{\partial y} = x \end{cases}$ $\begin{matrix} \Longrightarrow \frac{\partial}{\partial y}\frac{\partial F}{\partial x} = 2 \\ \Longrightarrow \frac{\partial}{\partial x}\frac{\partial F}{\partial y} = 1 \end{matrix}$

But! $\frac{\partial}{\partial x}\frac{\partial}{\partial y}F = \frac{\partial}{\partial y}\frac{\partial}{\partial x}F$ Equality of mixed partial derivatives

$$1 = 2$$

That is, if $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$

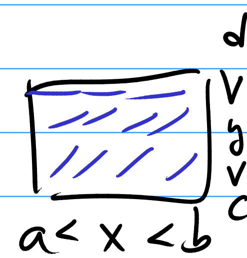
it follows necessarily that $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

If the function F exists, the equation is called <u>exact</u>.

exactness $\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

<u>Theorem</u>: If M, N, $\dfrac{\partial M}{\partial y}$, and $\dfrac{\partial N}{\partial x}$

are continuous in some rectangle



And $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Then there is a function F such that

$\dfrac{\partial F}{\partial x} = M \qquad \dfrac{\partial F}{\partial y} = N \qquad$ (it's exact!)

<u>Example</u>: $(y^2+2xy) + (2xy+x^2+3y^2)\dfrac{dy}{dx} = 0$

$M(x,y) = y^2+2xy$

$N(x,y) = 2xy+x^2+3y^2$

$\left[\dfrac{\partial M}{\partial y} = 2y+2x \qquad \dfrac{\partial N}{\partial x} = 2y+2x \right]$ ✓

How to solve: $\dfrac{\partial F}{\partial x} = y^2+2xy \qquad \dfrac{\partial F}{\partial y} = 2xy+x^2+3y^2$

Step: pick one equation and integrate it.

$$\frac{\partial F}{\partial x} = y^2 + 2xy$$

$$F = \int \frac{\partial F}{\partial x}\, dx = \int (y^2 + 2xy)\, dx$$

Treating y as a constant.

$$= y^2 x + x^2 y + C(y)$$

constant of integration with respect to x depends on y.

Now look at second equation

$$\frac{\partial F}{\partial y} = 2xy + x^2 + 3y^2$$

$$\frac{\partial}{\partial y}\left(y^2 x + x^2 y + C(y)\right) = 2xy + x^2 + 3y^2$$

$$\cancel{2xy} + \cancel{x^2} + \frac{\partial C}{\partial y}(y) = \cancel{2xy} + \cancel{x^2} + 3y^2$$

$$\frac{dC}{dy} = 3y^2$$

$$C(y) = \int \frac{dC}{dy}\, dy = \int 3y^2\, dy = y^3 + K$$

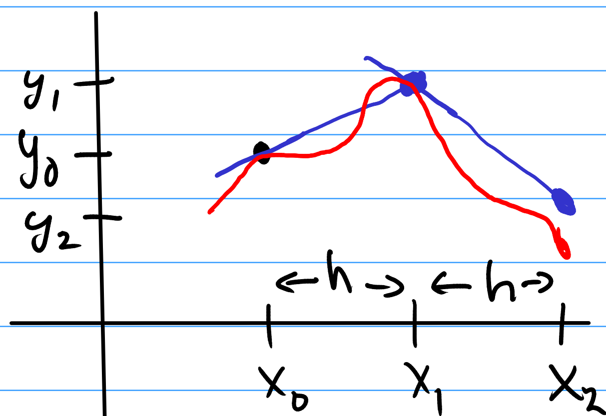
Put it all together $F(x,y) = x^2y + y^2x + y^3 + K$

<u>Notation</u>: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

is sometimes written $M(x,y)\,dx + N(x,y)\,dy = 0.$

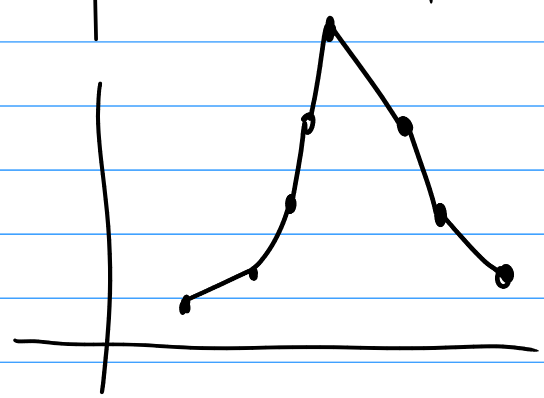
Euler's Method – Numerical Solution method

About initial value problem $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$

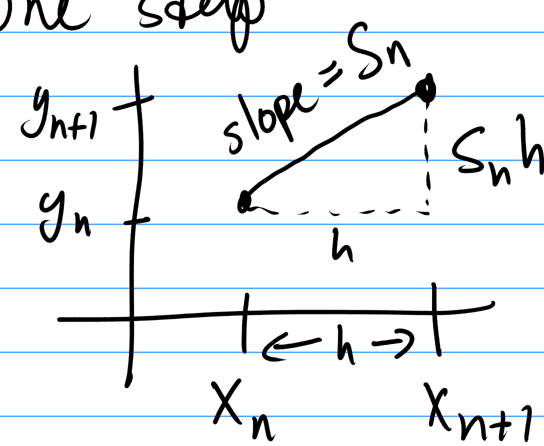


Broken line approximation

(Tangent line approximation)



One step



$$x_{n+1} - x_n = h$$

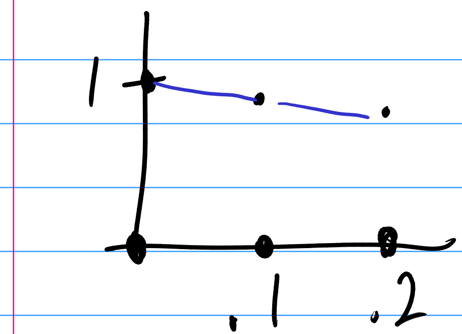
$$y_{n+1} - y_n = S_n h$$

$$S_n = f(x_n, y_n)$$

(where $y' = f(x,y)$)

$$y' = x^2 - y^2$$
$$y(0) = 1$$

Approximate $y(.2)$ using Euler's method with a step size $h = .1$

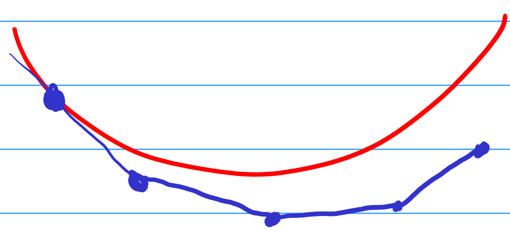


n	x_n	y_n	S_n	$S_n h$
0	0	1	-1	$-.1$
1	.1	.9	$-.80$	$-.08$
2	.2	.82		

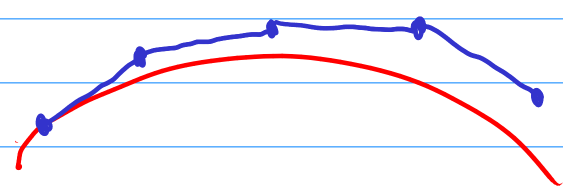
$$y(.2) \approx .82$$

Approximation gets better as h gets smaller

Suppose true looks like



Eulers method is too low



Euler's method is too high.