

# Autonomous equations

t independent variable  
y dependent variable

$$\frac{dy}{dt} = f(y)$$

Note: There is no t on the RHS.

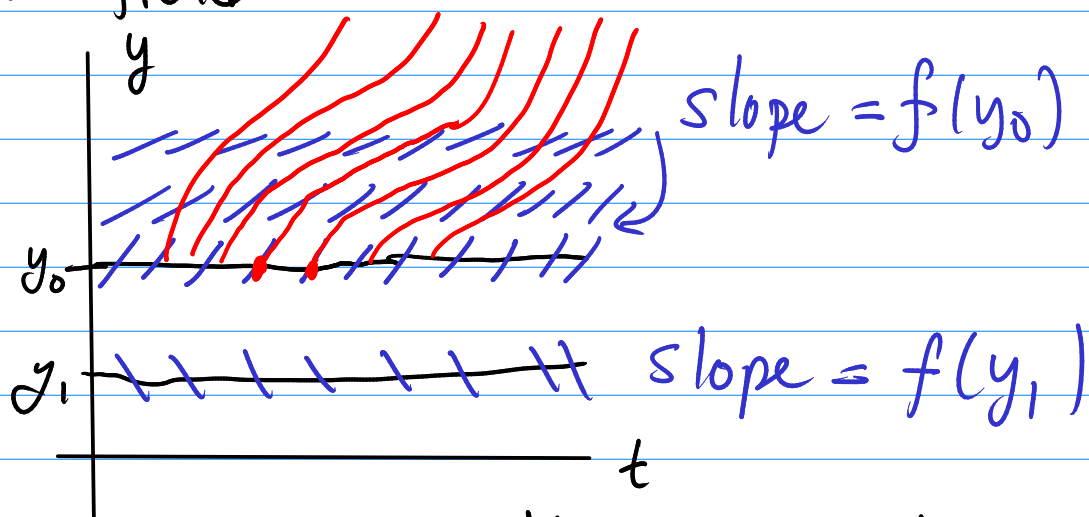
This is called autonomous equation.

(aka time-independent / time-invariant)

- Could solve this by separation of variables
- Can get a lot of qualitative info without explicitly solving the equation.

$$\frac{dy}{dt} = f(y)$$

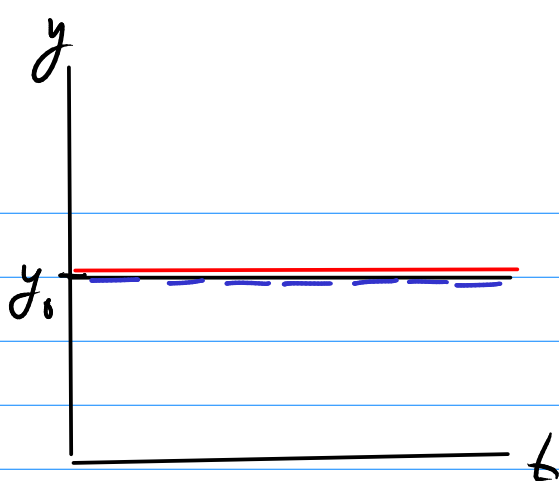
Direction field



- slope of direction field is constant along horizontal lines
- if we have one solution curve, shifting it to the left or right by any amount yields another solution curve.
- If we shift time axis, the equation doesn't change.
- If  $y(t)$  is a solution then so is  $y(t+c)$ .

Understanding solutions of  $\frac{dy}{dt} = f(y)$

Suppose that  $y_0$  is such that  $f(y_0) = 0$



$/$  = direction field  
 $-$  = solution curve

$$y(t) = y_0 \text{ (constant)}$$

Definition: a point  $y_0$  is called a critical point  
 (aka equilibrium point)  
 if any of the following equivalent conditions hold:

-  $f(y_0) = 0$

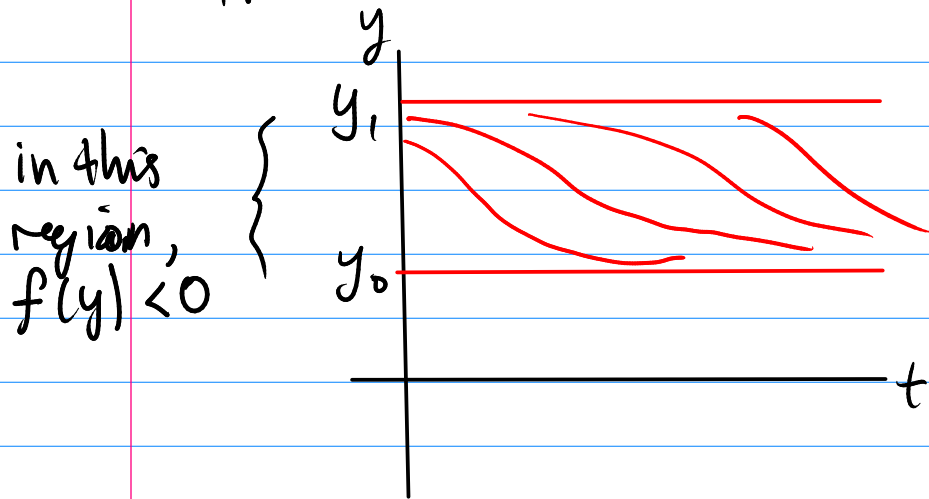
- slope of direction field at  $y_0 = 0$

-  $y(t) = y_0$  is a constant solution of the ODE

Proof that  $f(y_0) = 0 \Rightarrow y(t) = y_0$  is soluti.

$$\begin{array}{ccc}
 \frac{d}{dt} [y(t)] & \text{vs} & f(y(t)) \\
 \parallel & & \parallel \\
 \frac{d}{dt} (y_0) & & f(y_0) \\
 \parallel & & \parallel \\
 0 & & 0
 \end{array}$$

Suppose we have two critical points  $y_0, y_1$



other solutions  
cannot cross  
the critical points  
(constant solutions)

Step 1: find critical points  $f(y_0) = 0$

Step 2: what happens between critical points?

→ Draw the graph of  $f(y)$

→ figure out where  $f(y)$  is  $\begin{cases} \text{positive.} \\ \text{negative.} \end{cases}$

$\frac{dy}{dt} = f(y)$ : if  $f(y)$  is positive, then  $y$  is increasing  
if  $f(y)$  is negative, then  $y$  is decreasing.

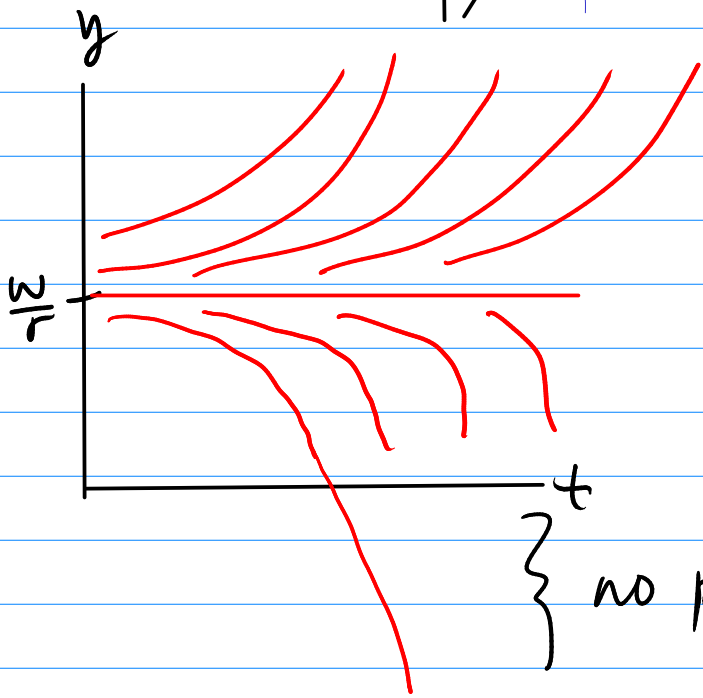
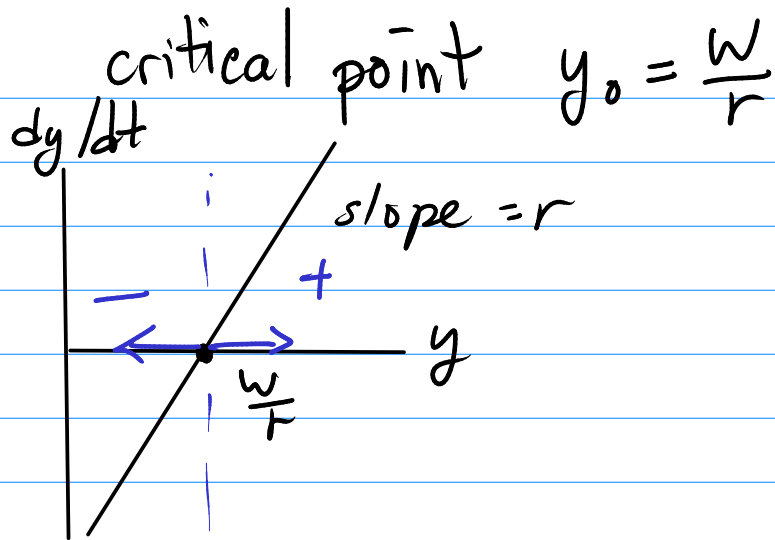
Example  $y$  = money in investment account

$r$  = interest rate (continuously compounded)

$$\frac{dy}{dt} = ry - w \quad \leftarrow \text{fees / theft}$$

$$\frac{dy}{dt} = ry - W$$

Graph  $f(y)$



A solution bigger than  $\frac{W}{r}$  increases

A sol<sup>n</sup> less than  $\frac{W}{r}$  decreases.

} no physical meaning.

Logistic equation (populations)  $y = \#$  in pop.

Natural growth  $\frac{dy}{dt} = ky$

→  $k$  is constant exponential growth.

→ logistic growth,  $k$  is not constant

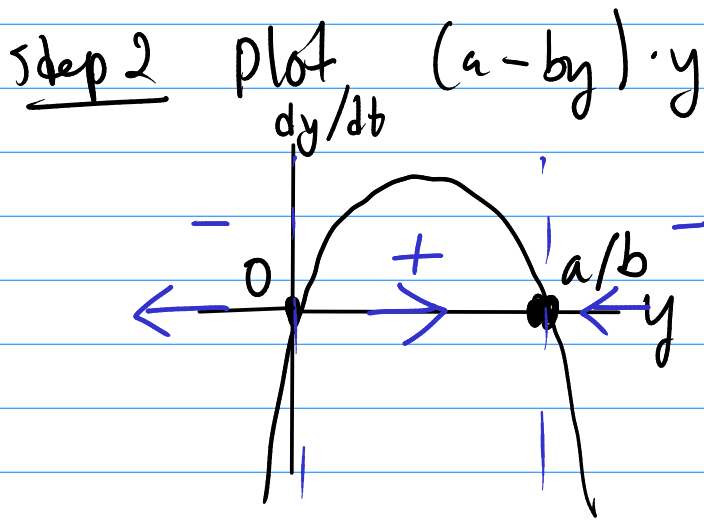
as  $y$  increases,  $k$  should decrease due limitations on food, shelter, etc

simplest possibility  $k = a - by$   
(linear  $a, b > 0$ )

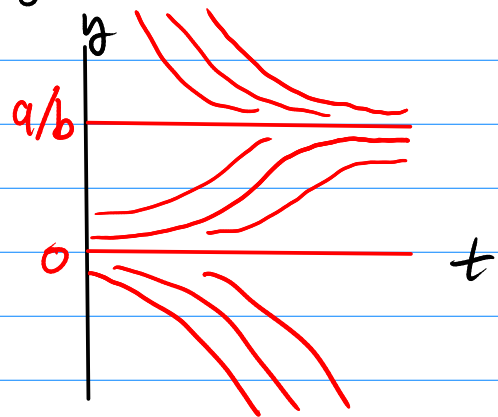
$$\frac{dy}{dt} = (a - by) \cdot y = ay - by^2 \quad (\text{Logistic equation})$$

Step 1 critical points  $(a - by_0) \cdot y_0 = 0$

$$\Rightarrow y_0 = \begin{cases} 0 \\ a/b \end{cases}$$



Solutions



Any solution where  $y > 0$  asymptotically stabilizes at  $a/b$

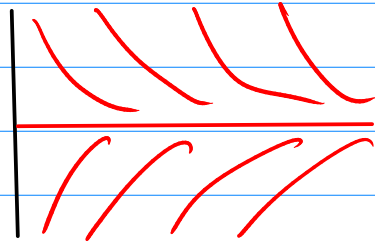
$a/b$  is called the "carrying capacity" in population examples.

$a/b$  is "attractive"

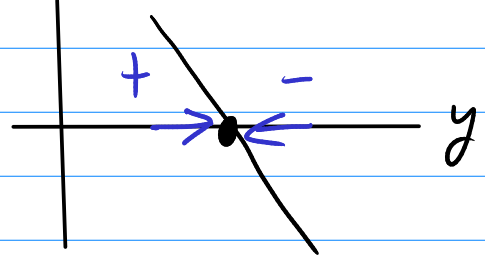
$0$  is "repulsive"

stable critical point  
stable equilibrium  
unstable critical point  
unstable equilibrium.

Stable

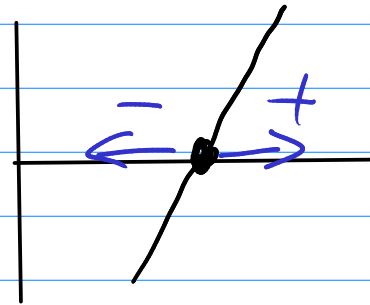
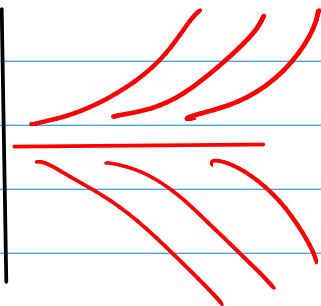


$dy/dt$



[restoring force or negative feedback loop]

Unstable

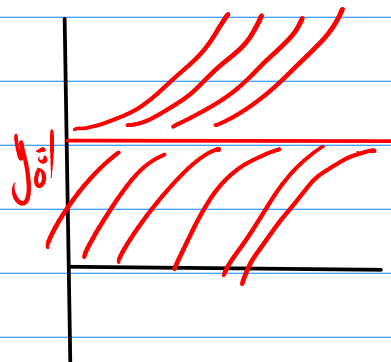
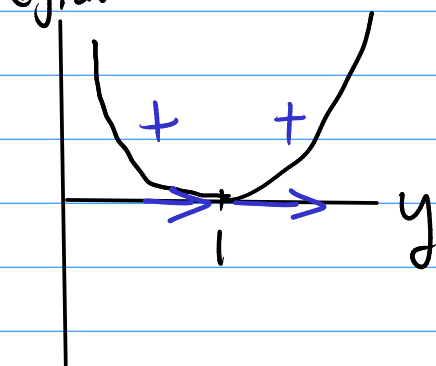


[amplifying force or positive feedback loop]

One more possibility:

$$f(y) = (y-1)^2$$

$y_0 = 1$  is the only critical point



Neither stable nor unstable: semistable.